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VOLUMEN SECUNDUM

IN QUO METHODUS INVENIENDI FUNCTIONES UNIUS VARIABILIS EX DATA RELATIONE DIFFERENTIALIUM SECUNDI ALTIORISVE GRADUS PERTRACTATUR.

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PETROPOLI,
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in Volumine secundo contentorum.

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Sectio secunda, de resolutione aequationum differentialium tertii altiorumque graduum, quae duas tantum variabiles involvunt.

CALCULI INTEGRALIS

LIBER PRIOR.

PARS SECUNDA,

SEU

METHODUS INVENIENDI FUNCTIONES UNIUS VARIABILIS EX DATA RELATIONE DIFFERENTIALIUM SECUNDI ALTIORISVE GRADUS.

SECTIO PRIMA,

DE

RESOLUTIONE AEQUATIONUM DIFFERENTIALIUM SECUNDI GRADUS DUAS TANTUM VARIABILES INVOLVENTIUM.

CAPUT I.

DE

INTEGRATIONE FORMULARUM DIFFERENTIALIUM SECUNDI GRADUS SIMPLICIUM.

Definitio.

706.

Positis binis variabilibus x et y, si vocetur $\partial y = p \partial x$ et $\partial p = q \partial x$, aequatio quaecunque, relationem inter quantitates x, y, p et q definiens, vocatur aequatio differentialis secundi gradus inter binas variabiles x et y.

Corollarium 1.

707. Quemadmodum ergo littera p implicat rationem differentialium primi gradus, dum est $p = \frac{\partial y}{\partial x}$, ita littera $q = \frac{\partial p}{\partial x}$ implicat rationem differentialium secundi gradus. Sumto enim ut vulgo fieri solet, elemento ∂x constante, erit $\partial p = \frac{\partial \partial y}{\partial x}$, ideoque $q = \frac{\partial \partial y}{\partial x^2}$.

Corollarium 2.

708. Quatenus ergo in aequatione proposita littera q inest, eatenus ea est differentialis secundi gradus. Si enim q abesset, ob solam p esset tantum differentialis primi gradus; ac si neque p neque q inesset, aequatio foret inter x et y, neque quicquam praeterea quaereretur.

Corollarium 3.

709. Methodus ergo desideratur, proposita aequatione quacunque praeter binas variabiles x et y, etiam quantitates $p = \frac{\partial y}{\partial x}$ et $q = \frac{\partial p}{\partial x}$ involvente, inveniendi relationem inter ipsas x et y, unde pateat, qualis y sit functio ipsius x, seu vicissim.

Scholion 1.

710. Hoc modo litteram q introducendo aequationes differentio-differentiales a conditione illa, qua quodpiam differentiale primi gradus pro constante assumi solet, liberantur. Cum enim ad meras quantitates finitas revocentur, quae rationem differentialium primi gradus exprimunt, consideratio differentialis constantis ne locum quidem habere potest. Quando ergo aequationes differentio-differentiales more solito ita exhibentur, ut quodpiam differentiale constans sit assumtum, introducendo litteras $p = \frac{\partial y}{\partial x}$ et $q = \frac{\partial p}{\partial x}$, species differentialium penitus tollitur, dum aequatio tantum quantitates finitas complectitur. Atque etiam vicissim proposita aequatione inter quantitates finitas x, y, p, q, ea ad formam vulgarem infinitis modis reduci potest, prout aliud atque aliud differentiale pro constante assumitur, quae tamen omnes formae specie diversae inter se perfecte conveniunt, quin etiam nullo differentiali constante assumto evolutio in formam solitam fieri potest.

Scholion 2.

711. Primum igitur breviter exponi conveniet, quomodo aequatio more solito per differentialia secundi gradus expressa ad formam nostram reduci queat, quodcunque differentiale constans fuerit assumtum. Sit ∂s hoc differentiale pro constante sumtum, cujus ergo ratio ad ∂x, ob $\frac{\partial y}{\partial x} = p$, per p et forte ipsas variabiles x et y datur; ponatur ergo $\partial s = v \partial x$, ut v fiat quantitas finita.

Jam cum in aequatione occurrant $\partial\partial x$ et $\partial\partial y$, vel alterutrum saltem, loco $\partial\partial x$ scribatur $\partial s \partial . \frac{\partial x}{\partial s}$, quia ob ∂s constans fit utique $\partial s . \partial . \frac{\partial x}{\partial s} = \partial\partial x$. Erit ergo $\partial\partial x = \partial s \partial . \frac{1}{v} = - \frac{\partial s \partial v}{v v}$. Simili modo loco $\partial\partial y$ scribendo $\partial s . \partial . \frac{\partial y}{\partial s} = \partial s \partial . \frac{p}{v}$, fiet $\partial\partial y = \frac{\partial s (v \partial p - p \partial v)}{v v}$. Cum igitur v per p, x et y detur, erit

$$\partial v = M \partial x + N \partial y + P \partial p = \partial x (M + N p + P q),$$

ob $\partial p = q \partial x$, sicque fiet

$$\partial\partial x = - \frac{\partial x^2}{v} (M + N p + P q) \text{ et}$$

$$\partial\partial y = \frac{\partial x^2}{v} (q v - M p - N p^2 - P p q),$$

hique valores loco $\partial\partial x$ et $\partial\partial y$ substituti in aequatione tantum differentialia primi gradus relinquent, quibus omnibus ad ∂x reductis, aequatio per divisionem prorsus a differentialibus liberabitur. Deinde vicissim hujusmodi aequatio inter x, y, p et r proposita in formam solitam, sumto quopiam elemento ∂s constante, evolvetur, si primo pro p ubique scribatur $\frac{\partial y}{\partial x}$, loco q autem $\frac{1}{\partial x} \partial . \frac{\partial y}{\partial x} = \frac{\partial x \partial\partial y - \partial y \partial\partial x}{\partial x^3}$ ubi quidem nullius adhuc elementi constantis ratio est habita. At ob $\partial s = v \partial x$ constans, insuper erit

$$v \partial\partial x + \partial v \partial x = 0, \text{ seu ob } \partial v = M \partial x + N \partial y + P \partial . \frac{\partial y}{\partial x},$$

$$v \partial\partial x + M \partial x^2 + N \partial x \partial y + \frac{P (\partial x \partial\partial y - \partial y \partial\partial x)}{\partial x} = 0,$$

unde pro lubitu vel $\partial\partial x$ vel $\partial\partial y$ elidi potest, neutro autem eliso infinitae formae aequivalentes exhiberi possunt.

Scholion 3.

712. Hinc ergo praestantia formae finitae, ad quam hic aequationes differentio-differentiales revocamus, prae more solito eas exhibendi luculenter perspicitur; cum eadem aequatio more solito infinitis modis, prout aliud atque aliud elementum constans assumitur, repraesentari possit, dum nostro more eadem aequatio semper ad unicam formam reducitur. Quodsi ergo nostro more aequatio-

nes prodeant diversae, certum est iis quoque diversas relationes, inter variabiles x et y exprimi, cum contra solito more diversissimae aequationes differentio-differentiales eandem relationem indicare queant, ex quibus plerumque difficile est eam eligere, quae ad resolutionem maxime sit accommodata. Cum igitur hic ejusmodi methodus requiratur, cujus ope proposita quacunque aequatione inter quaternas quantitates x, y, p et q, relatio inter binas variabiles x et y definiri queat, quoniam haec quaestio vires humanas superare videtur, a casibus simplicissimis erit exordiendum. Casus autem simplicissimi sine dubio sunt, quando in aequatione proposita duae tantum insunt quantitates, scilicet vel x et q tantum, vel y et q, vel p et q, hoc est si q aequetur functioni vel ipsius x, vel ipsius y, vel ipsius p tantum; quos casus in hoc capite evolvere constituimus.

Definitio.

713. Formula differentio-differentialis simplex est, quando posito $\partial y = p \partial x$ et $\partial p = q \partial x$, quantitas q aequatur functioni vel ipsius x, vel ipsius y; vel ipsius p tantum.

Corollarium 1.

714. Triplices ergo habemus formulas differentio-differentiales simplices, quarum resolutionem in hoc capite doceri convenit, prout quantitas q vel per functionem ipsius p, vel ipsius x, vel ipsius y tantum determinatur.

Corollarium 2.

715. Si ergo X denotet functionem ipsius x, Y ipsius y, et P ipsius p tantum, terna genera harum formularum simplicium sunt 1) $q = X$, 2) $q = Y$, 3) $q = P$; in quibus continetur casus simplicissimus $q = \text{Const.}$

Corollarium 3.

716. Si has formulas more solito exprimere velimus, ob $q = \frac{\partial p}{\partial x} = \frac{1}{\partial x} \partial . \frac{\partial y}{\partial x}$, sumto elemento ∂x constante, erit $q = \frac{\partial \partial y}{\partial x^2}$ sumto elemento ∂y constante, erit $q = - \frac{\partial y \partial \partial x}{\partial x^3}$; nullo autem sumto constante, erit $q = \frac{\partial x \partial \partial y - \partial y \partial \partial x}{\partial x^3}$, quibus simplicitas earum formularum haud mediocriter offuscatur.

Corollarium 4.

717. Si elementum $\sqrt{(\partial x^2 + \partial y^2)}$, quod saepe fit, constans accipiatur, erit $\partial x \partial \partial x + \partial y \partial \partial y = 0$; unde postremus valor ipsius q, vel ob $\partial \partial y = - \frac{\partial x \partial \partial x}{\partial y}$ abit in $q = - \frac{(\partial x^2 + \partial y^2) \partial \partial x}{\partial x^3 \partial y}$, vel ob $\partial \partial x = - \frac{\partial y \partial \partial y}{\partial x}$ abit in $q = \frac{(\partial x^2 + \partial y^2) \partial \partial y}{\partial x^4}$.

Scholion.

718. Repudiata ergo penitus vulgari ratione aequationes differentio-differentiales exprimendi, quippe qua formulae in se satis simplices vehementer complicatae evadere possent, ratione hic stabilita utamur, indeque resolutionem hujusmodi formularum simplicium doceamus.

Problema 92.

719. Posito $\partial y = p \partial x$ et $\partial p = q \partial x$, si q aequetur functioni cuicunque ipsius p, invenire relationem inter ipsas variabiles x et y.

Solutio.

Sit ergo $q = P$, denotante P functionem quamcunque ipsius p: quoniam igitur est $q = \frac{\partial p}{\partial x}$, erit $\partial p = P \partial x$, hincque

$$\partial x = \frac{\partial p}{P}, \text{ et } \partial y = p \partial x = \frac{p \partial p}{P}.$$

Ex quo consequimur integrando

$$x = a + \int \frac{\partial p}{P}, \text{ et } y = b + \int \frac{p \partial p}{P};$$

ita ut tam x, quam y per eandem novam variabilem p determinetur. Atque cum duae novae constantes a et b per duplicem integrationem sint introductae, hoc integrale pro completo erit habendum.

Corollarium 1.

720. Aequatio $q = P$, cujus integrationem hic tradimus, si in formam consuetam, sumto ∂x constante, resolvatur, ob $q = \frac{\partial \partial y}{\partial x^2}$, transmutabitur in $\partial \partial y = \partial x^2 f : \frac{\partial y}{\partial x}$; quae est aequatio differentio-differentialis, in qua ipsae variabiles x et y non occurrunt.

Corollarium 2.

721. Talis quoque forma prodit, si elementum ∂y vel alia expressio differentialis, in quam ipsae x et y non ingrediuntur, veluti $\sqrt{(\partial x^2 + \partial y^2)}$ pro constante sumatur. Hoc ergo modo omnis aequatio differentio-differentialis in quam ipsae variabiles x et y non ingrediuntur, integrari poterit.

Corollarium 3.

722. Sin autem hujusmodi elementum $y \partial x - x \partial y$ constans assumatur, ut $y \partial \partial x - x \partial \partial y = 0$, ob

$$q = \frac{1}{\partial x} \partial \cdot \frac{\partial y}{\partial x} = \frac{\partial x \partial \partial y - \partial y \partial \partial x}{\partial x^3},$$

$$q = \frac{(y \partial x - x \partial y) \partial \partial x}{x \partial x^3} = \frac{(y \partial x - x \partial y) \partial \partial y}{y \partial x^3},$$

quae expressio, si aequetur functioni ipsius $p = \frac{\partial x}{\partial y}$, integrari poterit.

Corollarium 4.

723. Si fuerit P quantitas constans, ut sit $q = f$, erit

$$x = a + \frac{p}{f} \text{ et } y = b + \frac{pp}{2f},$$

unde fit

$$y = b + \frac{f}{2}(x-a)^2, \text{ seu } y = \tfrac{1}{2}fxx - afx + \tfrac{1}{2}aaf + b,$$

seu mutata forma constantium $y = \frac{1}{2}fxx + Cx + D$.

Scholion.

724. Cum scilicet aequatio differentio-differentialis duplici integratione indigeat, si utraque omni extensione instituatur, duae novae constantes arbitrariae introducuntur; in quo criterium, num hujusmodi integrale sit completum, consistit. Quemadmodum enim aequationum differentialium primi gradus integratio completa unam constantem arbitrariam implicat, ita si aequatio differentialis fuerit secundi gradus, binae constantes novae in integrale completum ingredientur, ternae autem ac plures, si aequatio differentialis fuerit tertii altiorisve gradus. Problemata autem, quorum resolutio ad hujusmodi aequationes differentiales altiorum graduum deducunt, natura sua ita sunt comparata, ut solutionis determinatio totidem constantes requirat. Ita in aequatione $q = f$, seu sumto ∂x constante, $\partial\partial y = f\partial x^2$, aequatio integralis completa $y = \frac{1}{2}fxx + Cx + D$ duas constantes novas C et D involvit, quod etiam in subjunctis exemplis patebit.

Exemplum 1.

725. *Aequationis differentio-differentialis* $a\partial\partial y = \partial x \partial y$, *in qua elementum* ∂x *constans est sumtum, integrale completum invenire.*

Posito $\partial y = p\partial x$ et $\partial p = q\partial x$, erit $\partial\partial y = q\partial x^2$, hincque $aq = p$, et $P = \frac{p}{a}$. Quocirca integratio praebet

$$x = \int \frac{a\,\partial p}{p} = C + a\,lp \quad \text{et} \quad y = \int a\,\partial p = D + ap.$$

Cum igitur sit

$$p = \frac{y-D}{a}, \text{ erit } x = C + a\,l\frac{y-D}{a},$$

quae est aequatio integralis completa binas constantes C et D involvens.

Exemplum 2.

726. *Posito ∂x constante, invenire aequationem inter* x *et* y, *ut fiat* $\frac{(\partial x^2 + \partial y^2)\sqrt{(\partial x^2 + \partial y^2)}}{-\partial x \partial\partial y} = a$.

Posito $\partial y = p\partial x$, ob ∂x constans, erit $\partial\partial y = \partial p \partial x$, sicque nostra aequatio est $\frac{(1+pp)\sqrt{(1+pp)}}{-\partial p}\partial x = a$, unde fit

$$\partial x = \frac{-a\,\partial p}{(1+pp)^{\frac{3}{2}}} \quad \text{et} \quad \partial y = \frac{-a\,p\,\partial p}{(1+pp)^{\frac{3}{2}}}.$$

Per integrationem ergo nanciscimur

$$x = A - \frac{ap}{\sqrt{(1+pp)}}, \quad \text{et} \quad y = B + \frac{a}{\sqrt{(1+pp)}};$$

unde concludimus

$$(A - x)^2 + (y - B)^2 = aa.$$

Corollarium.

727. Si x et y denotent coordinatas rectangulas lineae curvae, formula $\frac{(\partial x^2 + \partial y^2)\sqrt{(\partial x^2 + \partial y^2)}}{-\partial x \partial\partial y}$ exprimit ejus radium osculi, qui ergo ut sit constans $= a$, aequatio integralis inventa circulum radio a describendum indicat.

Exemplum 3.

728. *Posito $\partial s = \sqrt{(\partial x^2 + \partial y^2)}$ eoque sumto constante, invenire aequationem inter* x *et* y, *ut fiat* $\frac{\partial s \partial y}{\partial\partial x} = \frac{a\,\partial x}{\partial y}$.

Ponatur $\partial y = p\,\partial x$, erit $\partial s = \partial x \sqrt{(1+pp)}$, et ob ∂s constans

$$\partial\partial x \sqrt{(1+pp)} + \frac{p\,\partial x\,\partial p}{\sqrt{(1+pp)}} = 0, \text{ seu}$$

$$\partial\partial x = \frac{-p\,\partial x\,\partial p}{1+pp},$$

unde aequatio proposita abit in

$$\frac{p\,\partial x^2 \sqrt{(1+pp)}}{-p\,\partial x\,\partial p}(1+pp) = \frac{a}{p}, \text{ seu}$$

$$\partial x = \frac{-a\,\partial p}{p(1+pp)^{\frac{3}{2}}}, \text{ et } \partial y = \frac{-a\,\partial p}{(1+pp)^{\frac{3}{2}}}, \text{ ergo}$$

$$y = D - \frac{ap}{\sqrt{(1+pp)}}.$$

At pro illa formula statuatur $p = \frac{1}{r}$, eritque

$$\partial x = \frac{arr\,\partial r}{(1+rr)^{\frac{3}{2}}} = \frac{a\,\partial r}{\sqrt{(1+rr)}} - \frac{a\,\partial r}{(1+rr)^{\frac{3}{2}}};$$

unde fit integrando

$$x = C - \frac{ar}{\sqrt{(1+rr)}} + a\,l[r + \sqrt{(1+rr)}], \text{ seu}$$

$$x = C - \frac{a}{\sqrt{(1+pp)}} + a\,l\frac{1+\sqrt{(1+pp)}}{p}.$$

Exemplum 4.

729. *Posito* $\partial s = \sqrt{(\partial x^2 + \partial y^2)}$, *hocque elemento sumto constante, fieri oporteat* $\frac{\partial s\,\partial y}{\partial\partial x} = a$ Ang. tang. $\frac{\partial y}{\partial x}$.

Si fiat ut ante $\partial y = p\,\partial x$, orietur haec aequatio integranda

$$\frac{-\partial x(1+pp)^{\frac{3}{2}}}{\partial p} = a \text{ Ang. tang. } p, \text{ seu}$$

$$\partial x = \frac{-a\,\partial p}{(1+pp)^{\frac{3}{2}}} \text{ Ang. tang. } p, \text{ et}$$

$$\partial y = \frac{-a p \partial p}{(1+pp)^{\frac{3}{2}}} \text{ Ang. tang. } p.$$

Cum nunc sit ∂ . Ang. tang. $p = \frac{\partial p}{1+pp}$, erit

$$x = \frac{-a p}{\sqrt{(1+pp)}} \text{ Ang. tang. } p - a \int \frac{p \partial p}{(1+pp)^{\frac{3}{2}}}, \text{ et}$$

$$y = \frac{a}{\sqrt{(1+pp)}} \text{ Ang. tang. } p - a \int \frac{\partial p}{(1+pp)^{\frac{3}{2}}}.$$

Quamobrem colligimus

$$x = C - \frac{ap}{\sqrt{(1+pp)}} - \frac{ap}{\sqrt{(1+pp)}} \text{ Ang. tang. } p, \text{ et}$$

$$y = D - \frac{a}{\sqrt{(1+pp)}} + \frac{a}{\sqrt{(1+pp)}} \text{ Ang. tang. } p.$$

Corollarium 1.

730. Si x sit abscissa et y applicata curvae, radius osculi proportionalis esse debet angulo, quem curvae tangens cum axe constituit; unde patet hanc curvam fore quandam spiralem, circa originem abscissarum se evolventem.

Corollarium 2.

731. Si angulus ille, cujus tangens $= p$, ponatur $= \Phi$, erit $p =$ tang. Φ, hincque

$$x = C - a \cos.\Phi - a\Phi \sin.\Phi, \text{ et}$$
$$y = D - a \sin.\Phi + a\Phi \cos.\Phi;$$

unde colligitur

$$x \cos.\Phi + y \sin.\Phi = C \cos.\Phi + D \sin.\Phi - a.$$

Corollarium 3.

732. Ut sumto $\Phi = 0$, ambae x et y evanescant, sumi debet $C = a$ et $D = 0$, eritque

$$x = a - a\cos.\Phi - a\Phi\sin.\Phi, \text{ et}$$

$$y = -a\sin.\Phi + a\Phi\cos.\Phi;$$

unde quamdiu angulus Φ est minimus, erit

$$x = -\tfrac{1}{2}a\Phi\Phi + \tfrac{1}{8}a\Phi^4 \text{ et } y = -\tfrac{1}{3}a\Phi^3 + \tfrac{1}{30}\Phi^5;$$

ideoque proxime

$$\frac{x^3}{yy} = -\tfrac{9}{8}a, \text{ seu } yy = -\frac{8x^3}{9a}.$$

Problema 93.

733. Posito $\partial y = p\partial x$ et $\partial p = q\partial x$, si quantitas q aequetur functioni ipsius x, quae sit X, definire relationem inter binas variabiles x et y.

Solutio.

Cum ergo sit $q = X$, erit $q\partial x = \partial p = X\partial x$, unde integrando colligimus $p = \int X\partial x + C$, atque hinc ob $\partial y = p\partial x$ adipiscemur

$$y = \int\partial x\int X\partial x + Cx + D.$$

At est

$$\int\partial x\int X\partial x = x\int X\partial x - \int Xx\partial x,$$

uti sumendis differentialibus sponte patet. Quare aequatio integralis completa relationem inter binas variabiles x et y continens est

$$y = x\int X\partial x - \int Xx\partial x + Cx + D$$

duas constantes arbitrarias C et D involvens. Quae ergo erit algebraica, si ambae formulae differentiales $X\partial x$ et $Xx\partial x$ integrationem admittant.

Corollarium 1.

734. Quodsi ergo sit $q = 0$, seu sumto ∂x constante, $\partial\partial y = 0$, ut sit $X = 0$, erit aequatio integralis completa $y = Cx + D$.

Corollarium 2.

735. Aequationes ergo differentio-differentiales, quas hoc modo integrare licet, sumto ∂x constante, in hac forma $\partial\partial y = X\partial x^2$ continentur, unde prima integratio praebet $\partial y = \partial x \int X \partial x + C$, et altera $y = \int \partial x \int X \partial x + Cx + D$.

Corollarium 3.

736. Sin autem differentiale ∂y capiatur constans ob $p = \frac{\partial y}{\partial x}$, erit $\partial p = -\frac{\partial y \partial\partial x}{\partial x^2} = q\partial x$, et forma aequationum hoc modo integrandarum erit $-\partial y \partial\partial x = X\partial x^3$.

Corollarium 4.

737. Quodsi elementum $\partial s = \sqrt{(\partial x^2 + \partial y^2)}$ sit constans, ob $\partial x \partial\partial x + \partial y \partial\partial y = 0$, erit

$$\partial p = \frac{\partial x \partial\partial y - \partial y \partial\partial x}{\partial x^2} = \frac{-\partial s^2 \partial\partial x}{\partial x^2 \partial y} = q\partial x.$$

Hinc forma aequationum hoc modo integrandarum est

$$-\partial s^2 \partial\partial x = X\partial x^3 \partial y.$$

Vel cum etiam sit

$$\partial p = q\partial x = +\frac{\partial s^2 \partial\partial y}{\partial x^3},$$

ea erit $\partial s^2 \partial\partial y = X\partial x^4$.

Scholion.

738. Hic manifestum est, quantum intersit aequationes differentio-differentiales a forma solita, ubi elementum quodpiam con-

stans est assumtum, ab hac conditione liberare et ad formam hic stabilitam reducere. Si enim proponatur haec aequatio $\partial s^2 \partial\partial y = X\partial x^4$, in qua elementum $\partial s = \surd(\partial x^2 + \partial y^2)$ constans sit assumtum, haud facile patet, quomodo ejus integratio sit suscipienda. Nostra autem methodo, si ponamus $\partial y = p\partial x$, ut sit

$$\partial s = \partial x \surd(1 + pp) \text{ et } \partial\partial y = p\partial\partial x + \partial x \partial p,$$

induit ista aequatio hanc formam

$$\partial x^2 (1 + pp)(p\partial\partial x + \partial x \partial p) = X\partial x^4 \text{ seu}$$

$$(p\partial\partial x + \partial x \partial p)(1 + pp) = X\partial x^2.$$

At quia ∂s, ac proinde quoque $\partial s^2 = \partial x^2 (1 + pp)$ est constans, erit

$$\partial\partial x (1 + pp) + p\partial x \partial p = 0, \text{ seu } \partial\partial x = \frac{-p\partial x \partial p}{1 + pp},$$

ideoque

$$p\partial\partial x + \partial x \partial p = \frac{\partial x \partial p}{1 + pp},$$

ita ut fiat $\partial p = X\partial x$, quae aequatio jam facillime tractatur. Hic scilicet in subsidium vocari debent ea, quae supra de integratione formularum differentialium simplicium sunt tradita.

Exemplum 1.

739. *Sumto ∂x constante, si fuerit $\partial\partial y = a x^n \partial x^2$, integrale completum investigare.*

Cum sit $\frac{\partial\partial y}{\partial x} = a x^n \partial x$, ob ∂x constans, erit integrando $\frac{\partial y}{\partial x} = \frac{a}{n+1} x^{n+1} + C$, hincque denuo integrando

$$y = \frac{a}{(n+1)(n+2)} x^{n+2} + Cx + D:$$

ubi casus $n = -1$ et $n = -2$ seorsim sunt evolvendi.

I. Ergo si $n = -1$, erit $\frac{\partial\partial x}{\partial x} = \frac{a\partial x}{x}$, hincque $\frac{\partial y}{\partial x} = a\,lx + C$, unde cum sit $\partial y = a\partial x\, lx + C\partial x$, erit denuo integrando $y = axlx -$

$\alpha x + C x + D$, seu loco $C - \alpha$ scribendo C, habebitur $y = \alpha x l x + C x + D$.

II. Si $n = -2$ et $\frac{\partial \partial y}{\partial x} = \frac{\alpha \partial x}{x^2}$, erit $\frac{\partial y}{\partial x} = \frac{-\alpha}{x} + C$, hincque $y = -\alpha l x + C x + D$.

Exemplum 2.

740. *Posito* $\partial s = \sqrt{(\partial x^2 + \partial y^2)}$ *constante, si fuerit*

$$\frac{\partial s^2 \partial \partial y}{\partial x^4} = \frac{1}{a} \cos. \frac{x}{c},$$

invenire integrale completum.

Ex superioribus constat fore $\frac{\partial s^2 \partial \partial y}{\partial x^4} = q$, ita ut proposita sit haec aequatio $q = \frac{1}{a} \cos. \frac{x}{c}$, unde fit

$$q \partial x = \partial p = \frac{\partial x}{a} \cos. \frac{x}{c}$$

et integrando

$$p = \frac{c}{a} \sin. \frac{x}{c} + C = \frac{\partial y}{\partial x}.$$

Quare obtinebitur

$$y = -\frac{cc}{a} \cos. \frac{x}{c} + C x + D,$$

quae est aequatio integralis completa.

Problema 94.

741. Posito $\partial y = p \partial x$ et $\partial p = q \partial x$, si quantitas q aequetur functioni cuicunque ipsius y tantum, quae sit Y, invenire aequationem integralem completam inter x et y.

Solutio.

Cum sit $q = Y = \frac{\partial p}{\partial x}$, erit $\partial x = \frac{\partial p}{Y}$, hincque $p \partial x = \partial y = \frac{p \partial p}{Y}$; unde conficitur haec aequatio inter p et y separata $p \partial p = Y \partial y$, quae integrata praebet

$$\tfrac{1}{2}pp = \int Y\partial y + \tfrac{1}{2}C \text{ et } p = \sqrt{(C + 2\int Y\partial y)} = \frac{\partial y}{\partial x}.$$

Hinc ergo porro concluditur $x = \int \frac{\partial y}{\sqrt{(C + 2\int Y\partial y)}}$, quae integratio denuo constantem arbitrariam inducit, ita ut hoc modo aequatio integralis completa inter x et y obtineatur.

Corollarium 1.

742. Cum sit $q = \frac{\partial p}{\partial x}$ et $\partial x = \frac{\partial y}{p}$, erit $q = \frac{p\partial p}{\partial y}$. Quare cum aequatio proposita sit $q = Y$, erit $\frac{p\partial p}{\partial y} = Y$, hincque $p\partial p = Y\partial y$, unde praecedens integratio sponte deducitur.

Corollarium 2.

743. Sumto elemento ∂x constante, cum sit $q = \frac{\partial\partial y}{\partial x^2}$, aequationes hic integratae habebunt hanc formam $\partial\partial y = Y\partial x^2$, cujus integratio, si per ∂y multiplicetur, est manifesta, fit enim

$$\tfrac{1}{2}\partial y^2 = \partial x^2 \int Y\partial y + \tfrac{1}{2}C\partial x^2,$$

ob ∂x constans, hincque $\partial x = \frac{\partial y}{\sqrt{(C + 2\int Y\partial y)}}$, ut ante.

Scholion.

744. En ergo specimen aequationum differentialium, quae per idoneum multiplicatorem integrabiles redduntur, ex quo intelligitur hanc methodum etiam in his aequationibus usum habere posse; deinceps autem locus erit hanc methodum uberius excolendi, cujus quippe usus praecipue in aequationibus differentialibus altiorum graduum est insignis, ubi variabilium separatio nihil subsidii affert. Atque hanc ob causam jam supra hanc methodum per multiplicatores integrandi commendavimus, alterique per separationem procedenti longe antetulimus.

Exemplum 1.

645. *Posito ∂x constante, si fuerit $aa\partial\partial y = y\partial x^2$ invenire integrale completum.*

Multiplicetur aequatio proposita per $2\partial y$, ut prodeat

$$2aa\partial y\partial\partial y = 2y\partial y\partial x^2,$$

quae ob ∂x constans habebit integrale

$$aa\partial y^2 = yy\partial x^2 + C\partial x^2,$$

unde colligitur

$$\partial x = \frac{a\partial y}{\sqrt{(yy+C)}},$$

quae denuo integrata dat

$$x = al[y + \sqrt{(yy + C)}] - alb,$$

unde concludimus, sumto e pro numero cujus logarithmus est $= 1$, fore

$$be^{\frac{x}{a}} = y + \sqrt{(yy + C)},$$

et irrationalitatem tollendo

$$bbe^{\frac{2x}{a}} - 2bye^{\frac{x}{a}} = C,$$

ita ut sit

$$y = \tfrac{1}{2}be^{\frac{x}{a}} - \frac{C}{2b}e^{-\frac{x}{a}}$$

at forma constantium C et b mutata, habebitur

$$y = Ce^{\frac{x}{a}} + De^{-\frac{x}{a}},$$

quae est aequatio integralis completa.

Exemplum 2.

746. *Posito ∂x constante, si fuerit $aa\partial\partial y + y\partial x^2 = 0$, invenire integrale completum.*

Multiplicatione per $2\partial y$ facta, aequationis

$$2aa\partial y\partial\partial y + 2y\partial y\partial x^2 = 0,$$

integrale est

$$aa\partial y^2 + yy\partial x^2 = cc\partial x^2,$$

unde deducimus

$$\partial x = \frac{a\partial y}{\sqrt{(cc-yy)}},$$

quae denuo integrata dat

$$x = a \text{ Ang. sin. } \frac{y}{c} + b.$$

Erit ergo

$$\frac{y}{c} = \sin.\frac{x-b}{a} = \cos.\frac{b}{a}\sin.\frac{x}{a} - \sin.\frac{b}{a}\cos.\frac{x}{a},$$

vel mutatis constantibus b et c, ita ut sit

$c\cos.\frac{b}{a} = C$ et $-c\sin.\frac{b}{a} = D$, erit

$$y = C\sin.\frac{x}{a} + D\cos.\frac{x}{a}.$$

Vel retenta prima forma, habemus.

$$y = C\sin.\left(\frac{x}{a} + \alpha\right).$$

Corollarium.

747. Hoc exemplum ex praecedente resolvi potuisset, cum sit

$$e^{u\sqrt{-1}} = \cos.u + \sqrt{-1}.\sin.u \text{ et } e^{-u\sqrt{-1}} = \cos.u - \sqrt{-1}.\sin.u,$$

ac vicissim

$$\cos.u\sqrt{-1} = \tfrac{1}{2}e^{u} + \tfrac{1}{2}e^{-u} \text{ et } \sin.u\sqrt{-1} = \frac{1}{2\sqrt{-1}}e^{u} - \frac{1}{2\sqrt{-1}}e^{-u}.$$

Exemplum 3.

748. *Posito ∂x constante, si fuerit $\partial\partial y\sqrt{ay} = \partial x^2$, integrale completum invenire.*

Cum ergo sit $2\partial y\partial\partial y = \frac{2\partial y}{\sqrt{ay}}.\partial x^2$,

erit integrando

$$\partial y^2 = \frac{4\partial x^2\sqrt{y}}{\sqrt{a}} + 4n\partial x^2 = \frac{4\partial x^2(\sqrt{y} + n\sqrt{a})}{\sqrt{a}},$$

unde colligimus

$$2\partial x = \frac{\partial y \sqrt[4]{a}}{\sqrt{(\sqrt{y}+n\sqrt{a})}}.$$

Sit commoditatis gratia $n\sqrt{a}=b$ et $\sqrt{y}=z$, ut fiat

$$\partial y = 2z\partial z \text{ et } \frac{\partial x\sqrt{n}}{\sqrt{b}} = \frac{z\partial z}{\sqrt{(b+z)}},$$

cujus integrale est

$$\frac{x\sqrt{n}}{\sqrt{b}} = \frac{2}{3}(z-2b)\sqrt{(b+z)}+C,$$

seu restituendo

$$\frac{x}{\sqrt[4]{a}} = \frac{2}{3}(\sqrt{y}-2\sqrt{c})\sqrt{(\sqrt{y}+\sqrt{c})}+C,$$

ubi c et C sunt binae constantes arbitrariae. Erit ergo

$$\frac{3(x+f)}{2\sqrt[4]{a}} = (\sqrt{y}-2\sqrt{c})\sqrt{(\sqrt{y}+\sqrt{c})},$$

posito $C=\frac{-f}{\sqrt[4]{a}}$, et sumtis quadratis

$$\frac{9(x+f)^2}{4\sqrt{a}} = y\sqrt{y}-3y\sqrt{c}+4c\sqrt{c}.$$

Scholion.

749. Forma ergo vulgaris aequationum hoc modo integrandarum, sumto elemento ∂x constante, est $\partial\partial y = Y\partial x^2$, quae per ∂y multiplicata manifesto fit integrabilis. Sin autem elementum ∂y capiatur constans, ob $q=\frac{\partial p}{\partial x}$ et $p=\frac{\partial y}{\partial x}$, erit $q=-\frac{\partial y\partial\partial x}{\partial x^3}$, hincque forma vulgaris $\partial y\partial\partial x = -Y\partial x^3$. Porro sumto elemento $\partial s=\sqrt{(\partial x^2+\partial y^2)}$ constante, ut fit $\partial x\partial\partial x+\partial y\partial\partial y=0$, ob $\partial p=\frac{\partial x\partial\partial y-\partial y\partial\partial x}{\partial x^2}$, erit vel $q=-\frac{\partial s^2\partial\partial x}{\partial x^3\partial y}$ vel $q=\frac{\partial s^2\partial\partial y}{\partial x^4}$; unde nascitur haec forma

$$-\frac{\partial s^2\partial\partial x}{\partial x^3\partial y}=Y \text{ vel } \frac{\partial s^2\partial\partial y}{\partial x^4}=Y,$$

quae etiam per ∂y multiplicatae integrabiles evadunt, etiamsi hoc jam minus pateat. Simili modo si elementum $y\partial x$ sumatur constans, ut sit $y\partial\partial x+\partial x\partial y=0$ et $\partial\partial x=-\frac{\partial x\partial y}{y}$ ob $\partial p=$

$\frac{\partial\partial y}{\partial x} + \frac{\partial y^2}{y\partial x}$, orietur haec forma $y\partial\partial y + \partial y^2 = Yy\partial x^2$, cujus membrum prius integrabile redditur, si per functionem quamcunque ipsarum $y\partial y$ et $y\partial x$ multiplicetur, ergo etiam per $\frac{y\partial y}{yy\partial x^2}$, quo multiplicatore simul alterum membrum $Yy\partial x^2$ redditur integrabile. His igitur casibus simplicissimis aequationum differentio-differentialium expeditis, qui ne ulla quidem difficultate laborant, ad difficiliores progrediamur, ac primo quidem ad eas aequationes, in quibus altera binarum variabilium x et y ipsa non inest; ita ut aequatio proposita ternas tantum contineat litteras x, p et q vel y, p et q, utriusque enim ratio fere perinde est comparata.

CAPUT II.

DE

AEQUATIONIBUS DIFFERENTIO-DIFFERENTIALIBUS IN QUIBUS ALTERA VARIABILIUM IPSA DEEST.

Problema 95.

750.

Posito $\partial y = p \partial x$ et $\partial p = q \partial x$, si detur aequatio quaecunque inter tres quantitates x, p et q, in quam altera variabilis y non ingrediatur, investigare relationem inter ipsas variabiles x et y.

Solutio.

Cum aequatio proposita has tres quantitates x, p et q contineat, loco q scribatur ejus valor $\frac{\partial p}{\partial x}$, atque habebitur aequatio differentialis primi gradus duas tantum quantitates variabiles x et p involvens, quam secundum praecepta prioris partis tractari, ejusque integrale investigari oportet. Integrali autem invento, quod si fuerit completum constantem arbitrariam complectetur, inde vel p per x, vel x per p determinari poterit. Priori casu quo p per x definire licet, ut p aequatur functioni cuidam ipsius x, quae sit $= X$, ob $p = X$ fiet $p \partial x = \partial y = X \partial x$, unde reperitur $y = \int X \partial x +$ Const. quae aequatio relationem desideratam inter x et y definit. Posteriori casu quo x per p detur, et functioni cuidam P ipsius p aequatur, ut sit $x = P$, erit $y = \int p \partial x = \int p \partial P$, seu $y = Pp - \int P \partial p$.

Sin autem neque x per p, neque p per x definiri queat, vi-

dendum est, num utramque per novam variabilem u exprimere liceat, unde fiat $x = V$ et $p = U$; tum enim habebitur $v = \int U \partial V$.

Corollarium 1.

751. Hujusmodi ergo aequationum differentio-differentialium resolutio ita instituitur, ut revocetur ad aequationem differentialem primi gradus inter binas variabiles x et p; quae si integrari queat, simul illius aequationis integratio habebitur, accedente quadam nova constante.

Corollarium 2.

752. Si aequatio inter x, p et q proposita ita fuerit comparata, ut q unicam dimensionem non excedat, vel si ad talem formam reduci patiatur, orietur aequatio differentialis simplex, differentialia unius tantum dimensionis involvens, ubi praecepta ante tradita in usum sunt vocanda.

Corollarium 3.

753. Sin autem quantitas q plures obtineat dimensiones, vel adeo transcendenter ingrediatur, tentanda sunt ea arficia, quae in fine superioris partis circa resolutionem hujusmodi aequationum sunt tradita.

Scholion.

754. Quando in aequatione inter x, p et q littera q unicam habet dimensionem, indeque posito $q = \frac{\partial p}{\partial x}$ aequatio differentialis simplex nascitur, praecipui casus, quibus integratio succedit, sunt: 1) si aequatio haec differentialis separationem admittat, 2) si alterutra variabilium p et x, differentialium quoque ratione habita, unam dimensionem non superet, ac 3) si ambae variabiles x et p ubique eundem dimensionum numerum constituant, quo casu aequatio ho-

mogenea appellatur. Casus minus late patentes, cujusmodi supra evolvimus, hic non commemoramus. Deinde si quantitas q vel pluribus dimensionibus sit implicata, vel adeo transcendenter ingrediatur, casus praecipui resolutionem admittentes, quemadmodum supra docuimus, sunt: 1) si proponatur aequatio quaecunque inter x et q deficiente p, 2) si aequatio tantum p et q contineat, quos binos quidem casus jam capite praecedente tractavimus; 3) si in aequatione proposita binae variabiles p et x ubique eundem dimensionum numerum constituant, 4) si in aequatione inter x, p et q altera binarum litterarum x vel p unicam dimensionem obtineat, denique 5) si aequatio ita fuerit comparata, ut posito $x = v^{\mu}$, $p = z^{\mu+\nu}$ et $q = t^{\nu}$, aequatio oriatur homogenea inter v, z et t, quae scilicet ubique eundem dimensionum numerum constituant. Secundum hos ergo casus exempla proferamus.

Exemplum 1.

755. *Investigare aequationem inter x et y, ut posita ∂x constante, haec formula $\frac{(\partial x^2 + \partial y^2)^{\frac{3}{2}}}{\partial x \partial \partial y}$ aequetur datae functioni ipsius x, quae sit* $= X$.

Posito $\partial y = p\partial x$ et $\partial p = q\partial x$, erit

$$\frac{(1+pp)^{\frac{3}{2}}}{q} = X = \frac{(1+pp)^{\frac{3}{2}}\partial x}{\partial p}, \text{ ideoque}$$

$$\frac{\partial x}{X} = \frac{\partial p}{(1+pp)^{\frac{3}{2}}},$$

ubi cum variabiles x et p sint a se invicem separatae. integratio dat

$$\frac{p}{\sqrt{(1+pp)}} = \int \frac{\partial x}{X}.$$

Ponatur $\int \frac{\partial x}{X} = V$, integrali completo sumto, erit V functio ipsius x, hinc

$$p = V\sqrt{(1+pp)} \text{ et } p = \frac{V}{\sqrt{(1-VV)}}.$$

Quare

$$\partial y = p\,\partial x = \frac{V\partial x}{\sqrt{(1-VV)}},$$

unde obtinetur

$$y = \int \frac{V\partial x}{\sqrt{(1-VV)}}.$$

Tum vero praeterea elicitur elementum

$$\sqrt{(\partial x^2 + \partial y^2)} = \partial x\sqrt{(1+pp)} = \frac{\partial x}{\sqrt{(1-VV)}},$$

cujus integrale praebet

$$\int \partial x\sqrt{(1+pp)} = \int \frac{\partial x}{\sqrt{(1-VV)}}.$$

Corollarium 1.

756. Si x et y sint coordinatae orthogonales curvae, erit formula $\frac{(1+pp)^{\frac{3}{2}}}{q}$ ejus radius curvedinis, unde hinc curva definitur, cujus radius curvedinis aequetur functioni cuicunque abscissae x.

Corollarium 2.

757. Si ergo radius curvedinis debeat esse reciproce proportionalis abscissae x, sumatur $X = \frac{aa}{2x}$, eritque

$$V = \int \frac{2x\partial x}{aa} = \frac{xx+ab}{aa}, \text{ hinc}$$

$$y = \int \frac{(xx+ab)\partial x}{\sqrt{[a^4-(xx+ab)^2]}},$$

quae conditio praebet curvas a lamina elastica formatas.

Corollarium 3.

758. Si fit $V = x^n$, seu $X = \frac{1}{n x^{n-1}}$ neglecta constante addenda, oritur $y = \int \frac{x^n \partial x}{\sqrt{(1 - x^{2n})}}$, quod integrale algebraice exhiberi potest casibus, quibus est vel $n = \frac{1}{2i+1}$ vel $n = \frac{-1}{2i}$, denotante i numerum integrum positivum.

Exemplum 2.

759. *Si posito ∂x constante, oporteat esse*

$$\partial x (\partial x^2 + \partial y^2) + x \partial y \partial \partial y = a \partial \partial y \sqrt{(\partial x^2 + \partial y^2)},$$

invenire aequationem inter x et y.

Posito $\partial y = p \partial x$, nostra aequatio ob $\partial \partial y = \partial p \partial x$ induit hanc formam

$$\partial x (1 + pp) + x p \partial p = a \partial p \sqrt{(1 + pp)},$$

quae per $\sqrt{(1 + pp)}$ divisa fit integrabilis, oritur enim

$$x \sqrt{(1 + pp)} = ap + b \text{ seu } x = \frac{ap + b}{\sqrt{(1 + pp)}}.$$

Cum nunc sit

$$y = \int p \partial x = px - \int x \partial p, \text{ erit}$$

$$y = \frac{app + bp}{\sqrt{(1 + pp)}} - \int \frac{\partial p (ap + b)}{\sqrt{(1 + pp)}},$$

et integratione evoluta

$$y = \frac{app + bp}{\sqrt{(1 + pp)}} - a \sqrt{(1 + pp)} - b\, l \frac{p + \sqrt{(1 + pp)}}{n}, \text{ seu}$$

$$y = \frac{bp - a}{\sqrt{(1 + pp)}} - b\, l \frac{p + \sqrt{(1 + pp)}}{n};$$

ita ut ambae variabiles x et y per p definiantur

Cum igitur ex priori eliciatur

$$p = \frac{ab + x\sqrt{(aa+bb-xx)}}{xx-aa} \text{ et } \sqrt{(1+pp)} = \frac{bx + a\sqrt{(aa+bb-xx)}}{xx-aa},$$

erit his valoribus substituis

$$y = \frac{a(aa+bb-xx)+bx\sqrt{(aa+bb-xx)}}{bx+a\sqrt{(aa+bb-xx)}} - bl\frac{b+\sqrt{(aa+bb-xx)}}{n(x-a)}, \text{ seu}$$

$$y = \sqrt{(aa+bb-xx)} - bl\frac{b+\sqrt{(aa+bb-xx)}}{n(x-a)}.$$

Corollarium.

760. Si constans priori integratione ingressa b evanescens sumatur, aequatio inter x et y fit algebraica, erit enim $y = \sqrt{(aa-xx)}$. Sin autem b non evanescat, aequatio integralis est transcendens, et logarithmos involvit.

Exemplum 3.

761. *Posito ∂x constante, si debeat esse*

$$aa\partial\partial y\sqrt{(aa+xx)} + aa\partial x\partial y = xx\partial x^2,$$

invenire aequationem inter x et y.

Posito $\partial y = p\partial x$, habebimus hanc aequationem

$$aa\partial p\sqrt{(aa+xx)} + aap\partial x = xx\partial x, \text{ seu}$$

$$\partial p + \frac{p\partial x}{\sqrt{(aa+xx)}} = \frac{xx\partial x}{aa\sqrt{(aa+xx)}},$$

in qua variabilis p unam dimensionem non superat. Cum ergo sit

$$\int\frac{\partial x}{\sqrt{(aa+xx)}} = l[x+\sqrt{(aa+xx)}],$$

haec aequatio integrabilis redditur, si multiplicetur per $x+\sqrt{(aa+xx)}$, tum enim prodit

$$p[x+\sqrt{(aa+xx)}] = \int\frac{xx\partial x[x+\sqrt{(aa+xx)}]}{aa\sqrt{(aa+xx)}}, \text{ seu}$$

$$p[x+\sqrt{(aa+xx)}] = \frac{1}{aa}\int\frac{x^3\partial x}{\sqrt{(aa+xx)}} + \frac{x^3}{3aa}, \text{ et}$$

$$\int\frac{x^3\partial x}{\sqrt{(aa+xx)}} = \tfrac{1}{3}(xx-2aa)\sqrt{(aa+xx)} + C,$$

hinc

$$p\,[x+\sqrt{(aa+xx)}] = \frac{(xx-2aa)\sqrt{(aa+xx)}+x^3}{3aa} + C.$$

Haec multiplicetur per $\sqrt{(aa+xx)} - x$, ut prodeat,

$$aap = \frac{-xx-2aa+2x\sqrt{(aa+xx)}}{3} + C\sqrt{(aa+xx)} - Cx,$$

et quia $\partial y = p\,\partial x$, erit integrando

$$aay = -\tfrac{1}{9}x^3 - \tfrac{2}{3}aax + \tfrac{2}{9}(aa+xx)\sqrt{(aa+xx)}$$
$$- \tfrac{1}{2}Cxx + C\int\partial x\sqrt{(aa+xx)}.$$

Quodsi ergo constans C evanescat, aequatio inter x et y erit algebraica, scilicet

$$9aay + 6aax + x^3 = 2(aa+xx)\sqrt{(aa+xx)}.$$

Exemplum 4.

762. *Posito ∂x constante, invenire integrale hujus aequationis differentio-differentialis*

$$(aa\,\partial y^2 + xx\,\partial x^2)\,\partial\partial y = nx\,\partial x^3\,\partial y.$$

Fiat $\partial y = p\,\partial x$, et ob $\partial\partial y = \partial p\,\partial x$ habebimus

$$(aapp+xx)\,\partial p = npx\,\partial x,$$

quae aequatio cum sit homogenea, statuamus $x = pu$, eritque

$$pp(aa+uu)\,\partial p = nppu(p\,\partial u + u\,\partial p),\ \text{seu}$$
$$\frac{\partial p}{p} = \frac{nu\,\partial u}{aa+(1-n)uu},$$

quae integrata dat

$$lp = \frac{n}{2(1-n)}\,l\,[aa+(1-n)uu] + \text{Const.}$$

Hinc colligitur

$$p = C\,[aa+(1-n)uu]^{\frac{n}{2(1-n)}},\ \text{atque}$$

$$x = Cu\,[aa+(1-n)uu]^{\frac{n}{2(1-n)}}.$$

Cum nunc sit

$$y = px - \int x\,\partial p\ \ \text{et}\ \ \partial p = Cnu\,\partial u\,[aa+(1-n)uu]^{\frac{3n-2}{2(1-n)}},$$

erit

$$y = CCu[aa + (1-n)uu]^{\frac{n}{1-n}} - nCC\int uu\partial u[aa + (1-n)uu]^{\frac{2n-1}{2(1-n)}}.$$

Casu autem $n = 1$ erit

$$lp = \frac{uu}{2aa} + C, \text{ et } u = a\sqrt{2l\frac{p}{c}}, \text{ hinc}$$

$$x = ap\sqrt{2l\frac{p}{c}}, \text{ et } y = app\sqrt{2l\frac{p}{c}} - a\int p\partial p\sqrt{2l\frac{p}{c}}.$$

Corollarium,

763. Si fuerit $n = \frac{1}{2}$, erit

$$x = Cu\sqrt{(aa + \tfrac{1}{2}uu)} \text{ et}$$

$$y = CCu(aa + \tfrac{1}{2}uu) - \frac{CCu^3}{6} + D = CCu(aa + \tfrac{1}{3}uu) + D;$$

sicque relatio inter x et y algebraice exprimitur, quod etiam fit; si $n = \frac{2}{3}$, vel $n = \frac{3}{4}$, vel $n = \frac{4}{5}$, etc.

Exemplum 5.

764. *Posito ∂x constante, integrare hanc aequationem differentio-differentialem*

$$a\partial x\partial y^2 + xx\partial x\partial\partial y = nx\partial y\sqrt{(\partial x^4 + aa\partial\partial y^2)},$$

Fiat $\partial y = p\partial x$ et $\partial p = q\partial x$, ut sit $\partial\partial y = q\partial x^2$, et nostra aequatio induet hanc formam

$$app + qxx = npx\sqrt{(1 + aaqq)},$$

quae est homogenea inter p et x. Statuatur ergo $p = ux$, fietque

$$auu + q = nu\sqrt{(1 + aaqq)}.$$

Jam vero est

$$\partial p = q\partial x = u\partial x + x\partial u,$$

unde fit $\frac{\partial x}{x} = \frac{\partial u}{q-u}$.

At ex illa aequatione inter q et u colligitur

$$q = \frac{auu + nu\sqrt{(1 - nn + aauu + a^4u^4)}}{nnaauu - 1}, \text{ et}$$

$$q - u = \frac{u(1 + au - nnaauu) + nu\sqrt{(1 - nnaauu + a^4u^4)}}{nnaauu - 1},$$

sicque

$$\frac{\partial x}{x} = \frac{\partial u}{u} \cdot \frac{nnaauu - 1}{1 + au - nnaauu + n\sqrt{(1 - nnaauu + a^4u^4)}},$$

Dabitur ergo x per u, hincque etiam $p = ux$ per u; unde deducitur $y = \int p\,\partial x = \int u.x\,\partial x$.

Corollarium 1.

765. Illa aequatio differentialis transformatur in hanc

$$\frac{\partial x}{x} = \frac{\partial u}{u} \cdot \frac{1 + au - nnaauu - n\sqrt{(1 - nnaauu + a^4u^4)}}{nn - 1 - 2au + (nn - 1)aauu},$$

unde ratio integrationis facilius perspicitur.

Corollarium 2.

766. Notatu dignus autem est casus $nn = 2$, quo fit

$$\frac{\partial x}{x} = \frac{\partial u}{u} \cdot \frac{1 + au - 2aauu - (1 - aauu)\sqrt{2}}{(1 - au)^2}, \text{ seu}$$

$$\frac{\partial x}{x} = \frac{\partial u}{u} \cdot \frac{1 + 2au - (1 + au)\sqrt{2}}{1 - au} = \frac{\partial u(1 - \sqrt{2})}{u} + \frac{a\partial u(3 - 2\sqrt{2})}{1 - au},$$

unde colligitur

$$lx = (1 - \sqrt{2})\,lu - (3 - 2\sqrt{2})\,l(1 - au) + \text{Const. seu}$$

$$xu^{\sqrt{2} - 1}(1 - au)^{3 - 2\sqrt{2}} = C.$$

Exemplum 6.

767. *Sumto elemento $\partial s = \sqrt{(\partial x^2 + \partial y^2)}$ constante, invenire integrale hujus aequationis*

$$\partial x^3 \partial y - x\partial s^2 \partial\partial y = a\partial x\partial s\sqrt{(\partial\partial x^2 + \partial\partial y^2)}.$$

Posito $\partial y = p\partial x$, erit $\partial s = \partial x\sqrt{(1 + pp)}$, et ob $\partial\partial s = 0$ fit

$$\partial\partial x = -\frac{p\partial p\partial x}{1 + pp} = \frac{-pq\partial x^2}{1 + pp},$$

existente $\partial p = q\partial x$, tum vero

$$\partial\partial y = p\partial\partial x + \partial p\partial x = \frac{-ppq\partial x^2}{1+pp} + q\partial x^2 = \frac{q\partial x^2}{1+pp},$$

ideoque

$$\surd(\partial\partial x^2 + \partial\partial y^2) = \frac{q\partial x^2}{\surd(1+pp)},$$

quibus substitutis, aequatio nostra induit hanc formam

$p - qx = aq$, quae differentiata praebet $-x\partial q = a\partial q$,

ideoque $\partial q = 0$ et $q = \frac{1}{c}$. Hinc $p = \int q\partial x = \frac{x+a}{c}$, qui idem valor ex aequatione $p = (x+a)q$ sine integratione obtinetur. Tum vero est $y = \int p\partial x = \frac{xx+2ax}{2c} + b$, quae est aequatio integralis completa binas constantes b et c involvens.

Exemplum 7.

768. *Sumto elemento $\partial s = \surd(\partial x^2 + \partial y^2)$ constante, invenire integrale hujus aequationis differentio-differentialis*

$$\partial x^3\partial y - x\partial s^2\partial\partial y = \frac{b\partial x^4\partial s^2\partial\partial x}{\surd(\partial x^6 + aa\partial s^4\partial\partial y^2)}.$$

Posito $\partial y = p\partial x$ et $\partial p = q\partial x$, ob $\partial s = \partial x\surd(1+pp)$ et $\partial\partial s = 0$, erit

$$\partial\partial x = \frac{-pq\partial x^2}{1+pp} \text{ et } \partial\partial y = \frac{-\partial x\partial\partial x}{\partial y} = \frac{-\partial\partial x}{p} = \frac{q\partial x^2}{1+pp},$$

ergo $\partial s^2\partial\partial y = q\partial x^4$, unde aequatio nostra fit

$$p - qx = \frac{bq}{\surd(1+aaqq)},$$

quae differentiata praebet

$$-x\partial q = \frac{b\partial q}{(1+aaqq)^{\frac{3}{2}}};$$

unde concluditur vel $\partial q = 0$ vel $x = \frac{-b}{(1+aaqq)^{\frac{3}{2}}}$.

Priori casu est $q = \frac{1}{c}$, et $p = \frac{x}{c} + \frac{b}{\surd(cc+aa)}$, hincque

$$y = \int p\partial x = \frac{xx}{2c} + \frac{bx}{\surd(cc+aa)} + f.$$

Posteriori casu quo $x = \frac{-b}{(1+aapq)^{\frac{3}{2}}}$ fit

$$p = \frac{-bq}{(1+aaqq)^{\frac{3}{2}}} + \frac{bq}{\sqrt{(1+aaqq)}} = \frac{aabq^3}{(1+aaqq)^{\frac{3}{2}}}.$$

At est

$$\partial x = \frac{+3aabq\partial q}{(1+aaqq)^{\frac{5}{2}}}, \text{ hincque}$$

$$\partial y = p\,\partial x = \frac{3a^4bbq^4\partial q}{(1+aaqq)^4},$$

et ope reductionum

$$y = \frac{-\frac{1}{2}bbq - aabbq^3}{(1+aaqq)^3} + \frac{1}{2}bb\int\frac{\partial q}{(1+aaqq)^3}.$$

Est vero

$$\int\frac{\partial q}{(1+aaqq)^{n+1}} = \frac{q}{2n(1+aaqq)^n} + \frac{2n-1}{2n}\int\frac{\partial q}{(1+aaqq)^n}.$$

Ergo

$$\int\frac{\partial q}{(1+aaqq)^3} = \frac{q}{4(1+aaqq)^2} + \frac{3}{4}\int\frac{\partial q}{(1+aaqq)^2} \text{ et}$$

$$\int\frac{\partial q}{(1+aaqq)^2} = \frac{q}{2(1+aaqq)} + \frac{1}{2}\int\frac{\partial q}{1+aaqq}$$

$$= \frac{q}{2(1+aaqq)} + \frac{1}{2a}\text{ Ang. tang. } aq.$$

Hinc

$$\int\frac{\partial q}{(1+aaqq)^3} = \frac{q}{4(1+aaqq)^4} + \frac{3q}{8(1+aaqq)} + \frac{3}{8a}\text{ Ang. tang. } aq,$$

ideoque

$$y = \frac{-bbq(1+2aaqq)}{2(1+aaqq)^3} + \frac{bbq}{8(1+aaqq)^2} + \frac{3bbq}{16(1+aaqq)}$$

$$+ \frac{3bb}{16a}\text{ Ang. tang. } aq,$$

existente

$$x = \frac{-b}{(1+aaqq)^{\frac{3}{2}}},$$

unde fit

$$1 + aaqq = \sqrt[3]{\frac{bb}{xx}},$$

ita ut hoc modo aequatio inter x et y exhiberi possit. Hoc autem integrale, ut supra vidimus, tantum est particulare.

Problema 96.

769. Posito $\partial y = p\partial x$ et $\partial p = q\partial x$, si detur aequatio quaecunque inter y, p et q, ita ut variabilis x ipsa in ea desit; investigare aequationem integralem inter x et y.

Solutio.

Cum sit $q = \frac{\partial p}{\partial x}$ et $\partial x = \frac{\partial y}{p}$, erit $q = \frac{p\partial p}{\partial y}$; in aequatione ergo inter y, p et q ubique loco q substituatur iste valor $\frac{p\partial p}{\partial y}$, atque habebitur aequatio differentialis primi gradus binas tantum variabiles p et y involvens, cujus resolutionem per methodos supra expositas tentari oportet. Inventa autem aequatione integrali inter p et y, inde vel p per y, vel y per p definiatur, quo facilius altera integratio institui possit. Si y per p commode definiri queat, ut y aequetur functioni cuipiam ipsius p, quae sit $= \mathrm{P}$, ut sit $y = \mathrm{P}$, erit $\partial x = \frac{\partial \mathrm{P}}{p}$, hincque $x = \int \frac{\partial \mathrm{P}}{p} = \frac{\mathrm{P}}{p} + \int \frac{\mathrm{P}\partial p}{pp}$. Sin autem commodius p per y definire liceat, ut sit $p = \mathrm{Y}$ denotante Y functionem quampiam ipsius y, ob $\partial x = \frac{\partial y}{p}$, habebitur $x = \int \frac{\partial y}{\mathrm{Y}}$. At si neutrum succedat, novam variabilem u introducendo, per eam utraque quantitas p et y definiatur, ut fiat $p = \mathrm{U}$ et $y = \mathrm{V}$, existentibus U et V functionibus ipsius u, atque hinc erit $\partial x = \frac{\partial \mathrm{V}}{\mathrm{U}}$, et $x = \int \frac{\partial \mathrm{V}}{\mathrm{U}}$; hocque modo per duplicem integrationem integrale completum obtinebitur.

Corollarium 1.

770. Hujusmodi ergo aequationum differentio-differentialium

resolutio quoque revocatur ad aequationem differentialem primi gradus, cujus resolutio si fuerit in potestate, simul illius integrale exhiberi poterit.

Corollarium 2.

771. Si aequatio inter y, p et q ita fuerit comparata, ut ex ea commode valor ipsius q elici queat, hincque q aequetur functioni ipsarum y et p, quae sit T, erit $p\partial p = T\partial y$, quae est aequatio differentialis primi gradus simplex.

Corollarium 3.

772. Sin autem hujusmodi evolutio non succedat, dum littera q vel ad altiores potestates exsurgit, vel signis radicalibus involvitur, vel adeo transcendenter ingreditur, aequatio differentialis quidem erit primi gradus sed complicata, quae methodis supra expositis erit tractanda.

Scholion 1.

773. Cum paucis casibus aequationes differentiales primi gradus integrari queant, eosdem etiam hic notasse et per exempla illustrasse juvabit. Interim vero et reliquos casus quasi solutos spectari convenit, quandoquidem in aequationibus differentialibus altiorum ordinum id potissimum desideratur ut earum resolutio ad ordinem inferiorem reducatur. Perpetuo enim in Analysi quae ordine tractationis praecedunt, tanquam penitus confecta spectari solent, etiamsi plurima adhuc desiderentur, ut hoc modo multitudo desideratorum diminuatur. Ita quamvis longe adhuc absit, quominus aequationes algebraicas omnium ordinum resolvere valeamus, dum adeo vires nostrae non ultra quartum extenduntur, tamen in Analysi sublimiori omnium istarum aequationum resolutionem pro cognita habemus. Quod etiam usu non caret, cum in praxi resolutio per approxima-

tionem, quam quousque luberit, extendere licet, sufficere possit. Simili modo etiam, quoniam methodum tradidimus, aequationum differentialium primi gradus integralia proxime inveniendi, merito totum negotium, ut plane confectum, est censendum, si eo resolutionem aequationum differentialium altiorum graduum reducere potuerimus. Quare in hac secunda parte statim atque aequationem differentialem secundi gradus ad primum gradum perduxerimus, totum negotium pro confecto erit habendum.

Scholion 2.

774. Aequationes ergo differentio-differentiales, quae hoc modo ad differentiales primi gradus reducuntur, ita sunt comparatae, ut posito $\partial y = p\partial x$ et $\partial p = q\partial x$, variabilis x ipsa inde tollatur, et aequatio inter solas tres variabiles y, p et q oriatur. Casus ergo quibus talis aequatio resolutionem admittit, duplicis sunt generis, ad quorum prius referendi sunt ii, quibus q unicam obtinet dimensionem, unde q functioni cuipiam ipsarum y et p aequari potest. Cum igitur sit $q = \frac{p\partial p}{\partial y} = f:(y \text{ et } p)$ quam ponamus $= T$, resolutio succedet. 1) Si T sit functio homogenea unius dimensionis ipsarum y et p. 2) Si fuerit $T = \frac{P}{y+Q}$, designantibus P et Q functiones quascunque ipsius p tantum, hinc enim fit

$$P\partial y = yp\partial p + Qp\partial p;$$

quorsum etiam refertur casus,

$$T = \frac{P}{y + Qy^n}.$$

3) Si fuerit $T = p(Yp + Z)$, si quidem Y et Z sint functiones quaecunque ipsius y, quia tum aequatio

$$\partial p = Yp\partial y + Y\partial y,$$

ob unicam dimensionem ipsius p est integrabilis, quorsum etiam referendus est casus $T = p(Yp + Zp^n)$. Pro altero genere si quan-

titas q plures habeat dimensiones, vel signis radicalibus sit implicata, vel adeo transcendenter ingrediatur, aequatio inter y, p et q, resolutionem admittet. 1) Si posito $q = pu$, ut sit $u = \frac{\partial p}{\partial y}$, aequatio resultet homogenea inter y et p, in qua scilicet y et p ubique eundem dimensionum numerum compleant, utcunque caeterum u in eam ingrediatur. 2) Si in aequatione post substitutionem $q = pu$ inter y, p et u orta, altera quantitas y vel p unicam obtineat dimensionem. 3) Si posito $y = v^{\mu}$, $p = z^{\mu+\nu}$ et $u = t^{\nu}$ aequatio oriatur homogenea inter ternas quantitas v, z et t, hujusmodi enim aequationes supra resolvere docuimus.

Exemplum 1.

775. *Posito elemento ∂x constante, si habeatur haec aequatio differentio-differentialis*

$$\partial\partial y + A\partial x\partial y + By\partial x^2 = 0,$$

ejus integrale completum invenire.

Posito $\partial y = p\partial x$ et $\partial p = q\partial x$, aequatio nostra erit

$$q + Ap + By = 0, \text{ seu } p\partial p + Ap\partial y + By\partial y = 0,$$

quae cum sit homogenea, posito $p = vy$, abit in

$$vvy\partial y + vyy\partial v + Avy\partial y + By\partial y = 0,$$

unde fit

$$\frac{\partial y}{y} + \frac{v\partial v}{vv + Av + B} = 0.$$

Sit $vv + Av + B = (v + \alpha)(v + \beta)$,

ut sit $\alpha + \beta = A$ et $\alpha\beta = B$ erit

$$\frac{\partial y}{y} + \frac{\alpha\partial v}{(\alpha - \beta)(v + \alpha)} - \frac{\beta\partial v}{(\alpha - \beta)(v + \beta)} = 0,$$

hincque integrando

$$ly + \frac{\alpha}{\alpha - \beta}l(v + \alpha) - \frac{\beta}{\alpha - \beta}l(v + \beta) = C \text{ seu}$$

$$y = a(v+\beta)^{\frac{\beta}{\alpha-\beta}}(v+\alpha)^{\frac{-\alpha}{\alpha-\beta}}, \text{ ideoque}$$
$$p = vy = av(v+\beta)^{\frac{\beta}{\alpha-\beta}}(v+\alpha)^{\frac{-\alpha}{\alpha-\beta}}.$$

Tum vero est

$$\partial x = \frac{\partial y}{p} = \frac{\partial y}{vy}, \text{ unde ob}$$
$$\frac{\partial y}{y} = \frac{-v\partial v}{vv+Av+B}, \text{ erit}$$
$$\partial x = \frac{-\partial v}{vv+Av+B} = \frac{\partial v}{(\alpha-\beta)(v+\alpha)} - \frac{\partial v}{(\alpha-\beta)(v+\beta)}, \text{ et}$$
$$x = \frac{1}{\alpha-\beta}\, l\, \frac{v+\alpha}{v+\beta} + \text{Const.}$$

Verum haec resolutio fit facilior sequenti modo:

Cum sit

$$\frac{\partial y}{y} = \frac{-v\partial v}{(v+\alpha)(v+\beta)} \text{ et } \partial x = \frac{-\partial v}{(v+\alpha)(v+\beta)}, \text{ erit}$$
$$\frac{\partial y}{y} + \alpha\partial x = \frac{-\partial v}{v+\beta} \text{ et } \frac{\partial y}{y} + \beta\partial x = \frac{-\partial v}{v+\alpha}, \text{ hinc}$$
$$ly + \alpha x = la - l(v+\beta) \text{ et } ly + \beta x = lb - l(v+\alpha).$$

Ergo

$$v+\beta = \frac{a}{y}e^{-\alpha x} \text{ et } v+\alpha = \frac{b}{y}e^{-\beta x},$$

unde fit

$$\alpha-\beta = \frac{1}{y}(be^{-\beta x} - ae^{-\alpha x}),$$

ideoque mutatis constantibus

$$y = \mathfrak{A}e^{-\alpha x} + \mathfrak{B}e^{-\beta x},$$

quae integratio locum habet, si α et β sint quantitates reales et inaequales. Cum igitur posuerimus

$$vv + Av + B = (v+\alpha)(v+\beta) \text{ erit}$$
$$\alpha = \tfrac{1}{2}A + \sqrt{(\tfrac{1}{4}AA - B)} \text{ et } \beta = \tfrac{1}{2}A - \sqrt{(\tfrac{1}{4}AA - B)},$$

hinc prout expressio $\frac{1}{4}AA - B$ fuerit vel positiva, vel negativa, vel evanescens, tres habebimus casus evolvendos:

1) Sit $\frac{1}{2}A = m$ et $\sqrt{(\frac{1}{4}AA - B)} = n$, erit aequationis propositae integrale completum

$$y = \mathfrak{A}e^{-(m+n)x} + \mathfrak{B}e^{-(m-n)x} = e^{-mx}(\mathfrak{A}e^{-nx} + \mathfrak{B}e^{nx}).$$

2) Sit $\frac{1}{2}A = m$ et $\sqrt{(\frac{1}{4}AA - B)} = n\sqrt{-1}$, ob

$e^{nx\sqrt{-1}} = \cos. nx + \sqrt{-1}. \sin. nx$ et

$e^{-nx\sqrt{-1}} = \cos. nx - \sqrt{-1}. \sin. nx$,

erit constantibus mutandis

$$y = e^{-mx}(\mathfrak{C}\cos. nx + \mathfrak{D}\sin. nx) = \mathfrak{E}e^{-mx}\cos.(nx + \mathfrak{e}).$$

3) Sit $\frac{1}{2}A = m$ et $\sqrt{(\frac{1}{4}AA - B)} = 0$, seu in casu primo $n = 0$, ob $e^{-nx} = 1 - nx$ et $e^{nx} = 1 + nx$ fiet

$$y = e^{-mx}(\mathfrak{C} + \mathfrak{D}x).$$

Corollarium 1.

776. Ad aequationis ergo propositae integrale inveniendum, aequationis $vv + Av + B = 0$ radices investigari oportet, quibus inventis facile erit integrale completum assignare.

Corollarium 2.

777. Haec autem aequatio quadratica $vv + Av + B = 0$ insignem habet analogiam cum ipsa aequatione proposita

$$\partial\partial y + A\partial y\partial x + By\partial x^2 = 0,$$

ex qua quippe oritur scribendo 1, v, v^2 loco y; $\frac{\partial y}{\partial x}$ et $\frac{\partial\partial y}{\partial x^2}$.

Corollarium 3.

778. Formata autem aequatione hac algebraica $vv + Av + B = 0$, si ejus factor sit $v + \alpha$, ex eo statim integrale particulare deducitur $y = \mathfrak{A}e^{-\alpha x}$, similiterque alter factor $v + \beta$ integrale parti-

culare dabit $y = \mathfrak{B} e^{-\beta x}$, quibus conjunctis obtinetur integrale completum $y = \mathfrak{A} e^{-\alpha x} + \mathfrak{B} e^{-\beta x}$.

Scholion.

779. Infra methodus facilior tradetur hujusmodi aequationes differentio-differentiales tractandi, quae adeo ad talem formam

$$\partial\partial y + P\partial y\partial x + Qy\partial x^2 = 0$$

patet, ubi P et Q sint functiones quaecunque ipsius x, quae etiam extendetur ad formam

$$\partial\partial y + P\partial y\partial x + Qy\partial x^2 = X\partial x^2,$$

sumendo pro X functionem quamcunque ipsius x. Methodus scilicet ea inde haurietur, quod in hujusmodi aequationibus variabilis y cum suis differentialibus ∂y et $\partial\partial y$ ubique unicam dimensionem constituat vel etiam nullam, ejusque ope resolutio ad aequationem differentialem primi gradus reducetur, quo ipso negotium pro confecto erit habendum. Quando autem hoc modo aequatio differentio-differentialis ad aequationem differentialem primi gradus reducitur, probe cavendum est, ne haec reductio pro integratione habeatur, quippe ad quam tantum ope idoneae substitutionis est perventum; nihilo enim minus duae adhuc integrationes supersunt absolvendae, quibus totidem constantes arbitrariae introducantur, si quidem integrale completum desideretur, quemadmodum in hoc exemplo et praecedentibus clare videmus.

Exemplum 2.

780. *Proposita aequatione differentio-differentiali*

$$ab\partial\partial y = \partial x\sqrt{(yy\partial x^2 + aa\partial y^2)},$$

ejus integrale investigare.

Posito $\partial y = p\partial x$ et $\partial p = q\partial x$, haec aequatio abit in hanc

$$abq = \surd(yy + aapp) = \frac{abp\partial p}{y}, \text{ ob } q = \frac{p\partial p}{\partial y},$$

quae cum sit homogenea ponatur $p = \frac{y}{u}$, erit

$$y\partial y\surd(1 + \frac{aa}{uu}) = \frac{aby}{u^3}(u\partial y - y\partial u), \text{ seu}$$

$$uu\partial y\surd(aa + uu) = abu\partial y - aby\partial u, \text{ unde fit}$$

$$\frac{\partial y}{y} = \frac{ab\partial u}{abu - uu\surd(aa + uu)}.$$

Ponatur $\surd(aa + uu) = su$, erit $uu = \frac{aa}{ss - 1}$,

$$\frac{\partial u}{u} = \frac{-s\partial s}{ss - 1}, \text{ et } \frac{\partial y}{y} = \frac{-bs\partial s}{bss - as - b} = \frac{-s\partial s}{ss - 2ns - s},$$

posito $\frac{a}{b} = 2n$. Ergo

$$\frac{2\partial y\surd(nn + 1)}{y} = \frac{-\partial s[n + \surd(nn + 1)]}{s - n - \surd(nn + 1)} + \frac{\partial s[n - \surd(nn + 1)]}{s - n + \surd(nn + 1)},$$

ideoque

$$y^{2\surd(nn+1)} = \frac{C[s - n + \surd(nn + 1)]^{n - \surd(nn+1)}}{[s - n - \surd(nn + 1)]^{n + \surd(nn+1)}}.$$

Datur igitur y per s, ut sit $y = S$, hincque

$$u = \frac{a}{\surd(ss - 1)} \text{ et } p = \frac{S\surd(ss - 1)}{a}, \text{ atque}$$

$$\partial x = \frac{a\partial S}{S\surd(ss - 1)}, \text{ seu } \partial x = \frac{-as\partial s}{(ss - 2ns - 1)\surd(ss - 1)},$$

quae formula ad rationalitatem perduci et per logarithmos seu arcus circulares integrare potest.

Exemplum 3.

781. *Posito* $\partial y = p\partial x$ et $\partial p = q\partial x$, *invenire integrale hujus aequationis* $\frac{(pp + yy)\surd(pp + yy)}{2pp + yy - qy} = ny$.

Cum sit $q = \frac{p\partial p}{\partial y}$, erit

$$\partial y(pp + yy)\surd(pp + yy) = 2nppy\partial y + ny^3\partial y - nyyp\partial p,$$

ob cujus homogeneitatem ponatur $p = uy$, fietque

$$y^3\partial y(uu + 1)^{\frac{3}{2}} = 2nuuy^3\partial y + ny^3\partial y - nu^2y^3\partial y - nuy^4\partial u$$

unde colligitur

$$\frac{\partial y}{y}=\frac{-nu\partial u}{(uu+1)\sqrt{(uu+1)}-nuu-n}=\frac{nu\partial u}{(uu+1)[n-\sqrt{(uu+1)}]},$$

et y per u definitur; ex quo erit $p=uy$ et

$$\partial x=\frac{\partial y}{uy}=\frac{n\partial u}{(uu+1)[n-\sqrt{(uu+1)}]}.$$

Casu quo $n=1$ erit

$$\frac{\partial y}{y}=\frac{-u\partial u}{(uu+1)[\sqrt{(uu+1)}-1]}=\frac{-\partial u[1+\sqrt{(uu+1)}]}{u(uu+1)},\ \text{et}$$

$$\partial x=\frac{-\partial u[1+\sqrt{(uu+1)}]}{uu(uu+1)}.\ \text{Est vero}$$

$$\int\frac{\partial u}{u(uu+1)}=l\frac{u}{\sqrt{(uu+1)}},\ \int\frac{\partial u}{uu(uu+1)}=-\frac{1}{u}-\text{Ang. tang. }u,$$

$$\int\frac{\partial u}{u\sqrt{(uu+1)}}=l\frac{\sqrt{(uu+1)}-1}{u},\ \int\frac{\partial u}{uu\sqrt{(uu+1)}}=-\frac{\sqrt{(uu+1)}}{u};$$

unde colligitur

$$y=\frac{C\sqrt{(uu+1)}}{\sqrt{(uu+1)}-1}=C\left(1+\frac{1}{\sqrt{(uu+1)}-1}\right)\ \text{et}$$

$$x=D+\frac{1+\sqrt{(uu+1)}}{u}+\text{Ang. tang. }u.$$

Inde est

$$\sqrt{(uu+1)}=\frac{y}{y-a};\ \text{et}\ u=\frac{\sqrt{(2ay-aa)}}{y-a},$$

ideoque

$$x=D+\sqrt{\frac{2y-a}{a}}+\text{Ang. cos. }\frac{y-a}{y},$$

quae formulae introducendo angulum Φ, cujus cosinus est $\frac{y-a}{a}$, ita commodius exhibentur

$$y=\frac{a}{1-\cos.\Phi}\ \text{et}\ x=\zeta+\Phi+\cot.\tfrac{1}{2}\Phi.$$

Corollarium 1.

782. Ex aequatione separata primum inventa solutio particularis eruitur, tribuendo ipsi u ejusmodi valorem constantem, ut

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denominator evanescat, qui est $u = \sqrt{(nn-1)}$; hinc $p = y\sqrt{(nn-1)}$ et $\partial x \sqrt{(nn-1)} = \frac{\partial y}{y}$, unde fit

$$ly = la + x\sqrt{(nn-1)}.$$

Corollarium 2.

783. Casu quo $n = 1$, hic casus particularis praebet $y = a$, pro valore quocunque alterius variabilis; fit enim $u = 0$, ideoque et $p = 0$, ita ut ex aequatione $\partial y = p\,\partial x$ quantitas x non determinetur.

Scholion.

784. Si y designet radium vectorem ex puncto fixo ad curvam quampiam ductum, et x angulum, quam iste radius cum recta quadam positione data constituit, formula

$$\frac{(pp+yy)\sqrt{(pp+yy)}}{2pp+yy-qy}$$

exprimit hujus curvae radium curvedinis. In exemplo ergo proposito ejusmodi quaeritur curva, cujus radius curvedinis aequetur ipsi ny, cui quaestioni casu $n = 1$ utique satisfacit valor $y = a$, qui praebet circulum, qui etiam ex aequatione integrali colligitur $y = \frac{C\sqrt{(uu+1)}}{\sqrt{(uu+1)}-1}$, sumendo constantem C nihilo aequalem, tum enim necesse est sit $u = 0$ et $p = 0$, sicque angulus x non determinatur. Praeter circulum autem infinitae aliae lineae curvae satisfaciunt. At si $n > 1$, solutio particularis $ly = la + x\sqrt{(nn-1)}$ praebet spiralem logarithmicam, praeter quam autem etiam infinitae aliae curvae satisfaciunt; casibus autem $n < 1$ nulla hujusmodi solutio particularis locum habet, sed formulas pro $\frac{\partial y}{y}$ et ∂x inventas revera integrari oportet.

Exemplum 4.

785. *Posito $\partial y = p\,\partial x$ et $\partial p = q\,\partial x$, invenire relationem inter x et y, ut fiat $\frac{(pp+yy)\sqrt{(pp+yy)}}{2pp+yy-qy} = a$.*

Cum sit $q = \frac{p\,\partial p}{\partial y}$, ponatur $pp + yy = zz$, ob $p\,\partial p = q\,\partial y$ erit $q\,\partial y + y\,\partial y = z\,\partial z$, seu $q + y = \frac{z\,\partial z}{\partial y}$. Aequatio autem proposita induit hanc formam:

$$z^3 = a\,(2\,zz - yy - qy) = a\,(2\,zz - \frac{yz\,\partial z}{\partial y}, \text{ seu}$$

$$zz\,\partial y = 2\,az\,\partial y - ay\,\partial z,$$

unde fit

$$\frac{\partial y}{y} = \frac{a\,\partial z}{2\,az - zz}, \text{ seu } \frac{2\,\partial y}{y} = \frac{\partial y}{z} + \frac{\partial z}{2\,a - z},$$

quare integrando colligitur

$$yy = \frac{Cz}{2\,a - z}, \text{ et } pp = zz - \frac{Cz}{2\,a - z} = \frac{-Cz + 2\,azz - z^3}{2\,a - z}.$$

At est $z = \frac{2\,ayy}{C + yy}$, ergo

$$pp = \frac{4\,aa\,y^4}{(C + yy)^2} - yy = \frac{yy\,[4\,aayy - (C + yy)^2]}{(C + yy)^2}.$$

Hinc igitur oritur

$$\partial x = \frac{(C + yy)\,\partial y}{y\sqrt{[4\,aayy - (C + yy)^2)}},$$

sit $yy = u$, erit

$$\partial x = \frac{(C + u)\,\partial u}{2\,u\sqrt{[4\,aau - (C + u)^2]}}.$$

Haec aequatio tractabilior redditur ponendo

$$u = 2\,aa - C + 2\,a\cos.\Phi\sqrt{(aa - C)},$$

fit enim

$$\partial x = \frac{-a\,\partial\Phi\,[a + \cos.\Phi\sqrt{(aa - C)}]}{2\,aa - C + 2\,a\cos.\Phi\sqrt{(aa - C)}}, \text{ seu}$$

$$2\,\partial x = -\partial\Phi - \frac{C\,\partial\Phi}{2\,aa - C + 2\,a\cos.\Phi\sqrt{(aa - C)}},$$

quae integrata dat

$$2\,x = \zeta - \Phi - \text{Ang. cos. } \frac{m + \cos.\Phi}{1 + m\cos.\Phi},$$

posito $m = \frac{2\,a\sqrt{(aa - C)}}{2\,aa - C}$, ut sit

$$C = \frac{2aa\sqrt{(1-mm)}}{1+\sqrt{(1-mm)}} \text{ et } \sqrt{(aa-C)} = \frac{ma}{1+\sqrt{(1-mm)}},$$

hincque $yy = \frac{2aa(1+m\cos.\Phi)}{1+\sqrt{(1-mm)}}$,

unde fit

$$\cos.\Phi = \frac{yy[1+\sqrt{(1-mm)}]-2aa}{2maa} \text{ et}$$

$$\frac{m+\cos.\Phi}{1+m\cos.\Phi} = \frac{yy[1+\sqrt{(1-mm)}]-2aa(1-mm)}{myy[1+\sqrt{(1-mm)}]}$$

Corollarium 1.

786. Cum sit $yy = \frac{2aa(1+m\cos.\Phi)}{1+\sqrt{(1-mm)}}$, erit

$$yy = aa + bb + 2ab\cos.\Phi,$$

si ponatur $b = \frac{a[1-\sqrt{(1-mm)}]}{m}$, unde fit

$$m = \frac{2ab}{aa+bb} \text{ et } \sqrt{(1-mm)} = \frac{aa-bb}{aa+bb},$$

hincque

$$2x = \zeta - \Phi - \text{Ang.}\cos.\frac{2ab+(aa+bb)\cos.\Phi}{aa+bb+2ab\cos.\Phi}, \text{ seu}$$

$$2x = \zeta - \Phi - \text{Ang.}\sin.\frac{(aa-bb)\sin.\Phi}{yy}.$$

Corollarium 2.

787. Si ut supra radius vector y cum angulo x referatur ad lineam curvam, hanc curvam circulum esse oportet radio $= a$ descriptum. Fit autem $\partial x = \frac{\partial\Phi(aa-ab\cos.\Phi)}{aa+bb-2ab\cos.\Phi}$ sumto $yy = aa+bb-2ab\cos.\Phi$, hincque

$$x = \zeta + \text{Ang.}\tang.\frac{a\sin.\Phi}{a\cos.\Phi - b},$$

cujus applicatio ad Geometriam rem facit perspicuam.

Exemplum 5.

788. *Sumto elemento ∂x constante, si proponatur haec aequatio $\partial\partial y(y\partial y + a\partial x) = \partial y(\partial x^2 + \partial y^2)$, ejus integrale invenire.*

Posito $\partial y = p\,\partial x$ et $\partial p = q\,\partial x$ habebimus

$$q(py + a) = p(1 + pp), \text{ et ob } q = \frac{p\,\partial p}{\partial y},$$
$$\partial p\,(py + a) = \partial y\,(1 + pp), \text{ sive}$$
$$\partial y - \frac{py\,\partial p}{1 + pp} = \frac{a\,\partial p}{1 + pp},$$

quae integrata dat

$$\frac{y}{\sqrt{(1 + pp)}} = \frac{ap}{\sqrt{(1 + pp)}} + b, \text{ ideoque}$$
$$y = ap + b\sqrt{(1 + pp)} \text{ et}$$
$$x = \int \frac{\partial y}{p} = a\,lp + b\,l\,[p + \sqrt{(1 + pp)}] + c,$$

ita ut x et y per eandem variabilem p exprimantur. Si constans b sumatur $= 0$, obtinetur integrale particulare

$$y = ap \text{ et } x = a\,lp + c = a\,l\frac{y}{a} + c,$$

seu in exponentialibus $y = c\,e^{x:a}$. Sin autem sumatur $b = a$, ob

$$p + \sqrt{(1 + pp)} = \frac{y}{a} \text{ et } p = \frac{yy - aa}{2ay}, \text{ erit}$$
$$x = a\,l\frac{yy - aa}{2aa} + c \text{ seu } yy = aa + c\,e^{x:a}.$$

Exemplum 6.

789. *Sumto ∂x constante, hujus aequationis differentio-differentialis*

$$\partial y^2 - y\,\partial\partial y = n\sqrt{(\partial x^2\,\partial y^2 + aa\,\partial\partial y^2)},$$

integrale invenire.

Posito $\partial y = p\,\partial x$ et $\partial p = q\,\partial x$, erit

$$pp - qy = n\sqrt{(pp + aaqq)},$$

quae facto $q = pu$, ut sit $\frac{p\,\partial p}{\partial y} = pu$ ideoque $\partial p = u\,\partial y$, abit in

$$pp - puy = np\sqrt{(1 + aauu)} \text{ seu}$$
$$p - uy = n\sqrt{(1 + aauu)}.$$

Jam quia $\partial p = u\,\partial y$ differentietur haec aequatio, prodibitque

$$-y\,\partial u = \frac{naau\,\partial u}{\sqrt{(1 + aauu)}},$$

hinc vel $\partial u = 0$ vel $y = \frac{-naau}{\sqrt{(1 + aauu)}}$.

1) Casu $\partial u = 0$ fit $u = \alpha$, $p = \alpha y + \beta$, et $\partial x = \frac{\partial y}{\alpha y + \beta}$, hinc $\alpha x = l(\alpha y + \beta) + C$.

2) Si $y = \frac{-naau}{\sqrt{(1 + aauu)}}$, erit

$$p = uy + n\sqrt{(1 + aauu)} = \frac{n}{\sqrt{(1 + aauu)}},$$

hincque

$$\partial x = \frac{\partial y}{p} = \frac{-aa\partial u}{1 + aauu} \text{ et } x = -a \text{ Ang. tang. } au + C,$$

vel ob $u = \frac{y}{a\sqrt{(nn.aa - yy)}}$, aequatio inter x et y quaesita erit

$$\frac{b - x}{a} = \text{Ang. tang.} \frac{y}{\sqrt{(nnaa - yy)}} = \text{Ang. sin.} \frac{y}{na},$$

unde fit $y = na \sin. \frac{b - x}{a}$. Haec autem relatio tantum pro integrali particulari est habenda.

CAPUT III.

DE AEQUATIONIBUS DIFFERENTIO-DIFFERENTIALIBUS HOMOGENEIS, ET QUAE AD EAM FORMAM REDUCI POSSUNT.

Problema 97.

790.

Aequationum differentio-differentialium homogenearum naturam explicare, atque ad formam finitam ponendo $\partial y = p \partial x$ et $\partial p = q \partial x$ accommodare.

Solutio.

Sumto elemento ∂x constante, aequatio differentio-differentialis vulgari modo expressa dicitur homogenea, si non solum ipsis variabilibus x et y, sed etiam earum differentialibus ∂x et ∂y, itemque ipsi $\partial \partial y$, singulis unam dimensionem tribuendo, omnes aequationis termini eundem dimensionum numerum contineant, veluti in hac aequatione

$$x x \partial \partial y + x \partial x^2 + y \partial y^2 = 0,$$

ubi in singulis terminis ternae insunt dimensiones. Quodsi ergo ponamus $\frac{\partial y}{\partial x} = p$, ac $\frac{\partial p}{\partial x} = \frac{\partial \partial y}{\partial x^2} = q$, littera p nullam dimensionem, littera vero q unam dimensionem negativam continere erit censenda. Hinc aequatio differentio-differentialis ad formam hic receptam reducta, ut nonnisi quantitates finitas x, y, p et q contineat, erit homogenea, si litteris x et y unam dimensionem tribuendo, litterae p vero nullam, at litterae q unam dimensionem negativam, in sin-

gulis aequationis terminis idem oriatur dimensionum numerus. Vicissim ergo quoties haec proprietas in aequatione inter quaternas quantitates x, y, p et q proposita deprehenditur, ea aequatio erit homogenea et forma vulgari expressa manifesto homogeneitatem prae se feret.

Corollarium 1.

791. Si ergo in aequatione tali homogenea inter x, y. p et q statuatur $y = u x$ et $q = \frac{v}{x}$, omnes termini eandem potestatem ipsius x continebunt, qua ergo per divisionem sublata, aequatio prodibit tres tantum variabiles u, v et p involvens.

Corollarium 2.

792. Criterium igitur aequationis homogeneae inter quatuor quantitates x, y, p et q propositae in hoc consistit, ut posito $y = u x$ et $q = \frac{v}{x}$, quantitas x prorsus ex calculo exterminetur.

Corollarium 3.

793. Facta itaque hac substitutione, qua obtinetur aequatio inter ternas quantitates u, v et p, ex ea pro lubitu vel p per u et v, vel v per u et p, vel u per v et p definiri poterit.

Scholion.

794. Simili modo ideam homogeneitatis in aequationibus differentio-differentialibus constituimus, quo in aequationibus differentialibus primi gradus sumus usi. In his quidem, cum differentialia sponte eundem dimensionum numerum constituere debeant, homogeneitas ex solis ipsis variabilibus x et y dijudicatur. At in aequationibus differentio-differentialibus praeter ipsas variabiles x et y etiam litterae q ratio in computo dimensionum haberi debet, ita ta-

men ut ipsi una dimensio negativa sit tribuenda; littera autem p in hunc computum plane non ingreditur, quae ergo utcunque aequationi implicetur, homogeneitatem non turbat. Plurimum autem interest probe nosse indolem differentio-differentialium aequationum homogenearum, cum earum resolutio ad resolutionem aequationum differentialium primi gradus reduci possit, ita ut si haec successerit, etiam ipsarum aequationum differentio-differentialium integratio habeatur, id quod in sequenti problemate luculentius ostendemus.

Problema 98.

795. Proposita aequatione differentio-differentiali homogenea, ejus resolutionem ad integrationem aequationis differentialis primi gradus reducere.

Solutio.

Reducta aequatione ponendo $\partial y = p \partial x$ et $\partial p = q \partial x$ ad formam hic receptam, ut habeatur aequatio inter quatuor quantitates finitas x, y, p et q, ponatur $y = u x$ et $q = \frac{v}{x}$, ac cum aequatio sit homogenea, hoc modo quantitas x penitus ex calculo elidetur, ita ut proditura sit aequatio inter ternas quantitates u, v et p, ex qua unam per binas reliquas definire liceat. Nunc igitur cum sit $\partial y = p \partial x$, erit $u \partial x + x \partial u = p \partial x$, hincque $\frac{\partial x}{x} = \frac{\partial u}{p - u}$. Deinde ob $\partial p = q \partial x$, erit $\partial p = \frac{v \partial x}{x}$, ideoque $\frac{\partial x}{x} = \frac{\partial p}{v}$, ex quo duplici ipsius $\frac{\partial x}{x}$ valore colligitur $\frac{\partial u}{p - u} = \frac{\partial p}{v}$, seu $v \partial u = p \partial p - u \partial p$. Quodsi ergo ex illa aequatione quantitas v definiatur per binas p et u, habebitur aequatio differentialis primi gradus inter binas variabiles p et u, cujus integratio si fuerit in potestate, ut p per u innotescat, aequatio $\frac{\partial x}{x} = \frac{\partial u}{p - u}$, in qua variabiles x et u sunt separatae, integretur, sicque x per u definietur; unde fit $y = u x$; seu statim in hoc integrali loco u scribatur $\frac{y}{x}$, et habebitur aequatio inter x et y quaesita.

Corollarium 1.

796. Totum ergo negotium reducitur ad integrationem hujus aequationis differentialis simplicis $v \partial u = p \partial p - u \partial p$, quae si ope regularum supra traditarum expediri queat, simul aequationis differentio-differentialis integratio habetur.

Corollarium 2.

797. Simul autem patet resolutionem hujusmodi aequationum duplicem integrationem requirere, unde duae quantitates arbitrariae constantes ingredientur, quibus integrale completum constituitur.

Corollarium 3.

798. Etiamsi autem integratio $v \partial u = p \partial p - u \partial p$ non succedat, tamen ingens lucrum est rem eo perduxisse, cum supra methodus generalis sit tradita integralia omnium aequationum differentialium primi gradus proxime assignandi.

Scholion.

799. Operae igitur pretium erit eos casus perpendere, quibus aequatio $v \partial u = p \partial p - u \partial p$ integrationem admittit; quamobrem examinemus, qualis functio v debeat esse ipsarum p et u, ut hoc eveniat. Primum autem patet hoc fieri, si v fuerit functio homogenea unius dimensionis ipsarum p et u, quoniam tum ipsa haec aequatio fit homogenea ac per regulas supra expositas ad integrationem perduci potest. Deinde etiam integratio succedit si fuerit v functio quaecunque ipsius p, quoniam tum altera variabilis u unam dimensionem non superat, et aequationis $\partial u + \frac{u \partial p}{v} = \frac{p \partial p}{v}$ integrale est

$$e^{\int \frac{\partial p}{v}} u = \int \frac{e^{\int \frac{\partial p}{v}} p \partial p}{v}.$$

Tertio integrationem absolvere licebit, si v fuerit functio quaecunque quantitatis $p-u$. Posito enim $p-u=s$, ut sit v functio ipsius s, ob $p=s+u$, nostra aequatio erit $v\partial u=s\partial s+s\partial u$, ideoque $\partial u=\frac{s\partial s}{v-s}$ et $u=\int\frac{s\partial s}{v-s}$, quae integratio adeo ad formulas simplices est referenda. Quarto manente $s=p-u$, si P, Q, R denotent functiones quascunque ipsius s, aequatio nostra $v\partial u=s\partial s+s\partial u$ tractari poterit, si fuerit $v=s+\frac{Ps}{Qu+Ru^n}$, tum enim fit $P\partial u=Qu\partial s+Ru^n\partial s$. Quinto etiam patet, si denotantibus V et U functiones quascunque ipsius u, fuerit $v=s+Vss+Us^n$, integrationem quoque fore in potestate, fit enim aequatio nostra $Vs\partial u+Us^{n-1}\partial u=\partial s$. Atque in genere si aequatio differentialis $\partial s=Z\partial u$ fuerit integrabilis, existente Z functione binarum variabilium s et u, cum nostra aequatio sit $s\partial s=(v-s)\partial u$, habebimus $v=s+Zs$ pro omnibus casibus integrationem admittentibus.

Exemplum 1.

800. *Sumto elemento ∂x constante, si proponatur haec aequatio $xx\partial\partial y=x\partial x\partial y+ny\partial x^2$, ejus integrale invenire.*

Posito $\partial y=p\partial x$ et $\partial p=q\partial x$, erit $qxx=px+ny$, unde facto $y=ux$, prodit $qx=p+nu=v$, ita ut v sit functio unius dimensionis ipsarum p et u, et aequatio nostra $(p+nu)\partial u=p\partial p-u\partial p$ fiat homogenea. Cum ergo sit $nu\partial u+p\partial u+u\partial p=p\partial p$, erit integrando

$$C+nuu+2pu=pp \text{ et}$$

$$p=u+\sqrt{[C+(n+1)uu]}.$$

Habebimus ergo $\frac{\partial x}{x}=\frac{\partial u}{\sqrt{[C+(n+1)uu]}}$, quae denuo integrata dat

$$lx=\frac{1}{\sqrt{(n+1)}}\,l\,\frac{u\sqrt{(n+1)}+\sqrt{[C+(n+1)uu]}}{D}, \text{ seu}$$

$$D\,x^{\sqrt{(n+1)}} = u\sqrt{(n+1)} + \sqrt{[C + (n+1)\,u\,u]}$$

hincque

$$D^2\,x^{2\sqrt{(n+1)}} - 2\,D\,x^{\sqrt{(n+1)}}\,u\sqrt{(n+1)} = C:$$

Sit $D = f\sqrt{(n+1)}$ et $C = g\,(n+1)$

ut habeatur

$$ff\,x^{2\sqrt{(n+1)}} - 2\,f\,x^{\sqrt{(n+1)}}\,u = g, \text{ existente } u = \frac{y}{x}.$$

Casu quo $u = -1$, ob $\frac{\partial x}{x} = \frac{\partial u}{\alpha}$, erit $\alpha\,l\frac{x}{a} = u = \frac{y}{x}$, ideoque $y = \alpha\,x\,l\frac{x}{a}$. At si $n+1$ sit numerus negativus integratio etiam angulos implicabit.

Corollarium 1.

801. Si sit $n = 0$, hujus aequationis $xx\partial\partial y = x\partial x\partial y$ integrale completum erit $ff\,x^2 - 2\,f\,y = g$, qui casus per se est perspicuus. Cum ex $\frac{\partial\partial y}{\partial y} = \frac{\partial x}{x}$ fluat

$$\frac{\partial y}{\partial x} = f\,x \text{ et } 2\,y = f\,x\,x - \frac{g}{f}.$$

Corollarium 2.

802. Si sit $n = 3$, aequationis $x\,x\,\partial\,\partial\,y = x\,\partial\,x\,\partial\,y + 3\,y\,\partial\,x^2$ integralum completum est $ff\,x^4 - 2\,f\,x\,y = g$. Idem evenit si loco $\sqrt{(n+1)}$ scribatur -2, fit enim

$$\frac{ff}{x^4} - \frac{2fy}{x^3} = g \text{ et } ff - 2\,f\,x\,y = g\,x^4,$$

utraque redit ad $y = \frac{\alpha}{x} + \beta\,x^3$.

Exemplum 2.

803. *Sumto elemento $\partial\,x$ constante, si proponatur haec aequatio $\frac{xx\partial\partial y}{\partial x} = \sqrt{(m\,x\,x\,\partial\,y^2 + n\,y\,y\,\partial\,x^2)}$, ejus integrale completum invenire.*

Ob $\partial y = p \partial x$ et $\partial p = q \partial x$, habebimus $q x x = \sqrt{(m p p x x + n y y)}$, quae posito $y = u x$ abit in

$$q x = \sqrt{(m p p + n u u)} = v, \text{ ob } q = \frac{v}{x}.$$

Quia ergo est $\frac{\partial x}{x} = \frac{\partial u}{p - u} = \frac{\partial p}{v}$, erit

$$\partial u \sqrt{(m p p + n u u)} = (p - u) \partial p,$$

quae est aequatio homogenea. Ponatur ergo $p = s u$, et prodit

$$\partial u \sqrt{(m s s + n)} = (s - 1)(s \partial u + u \partial s);$$

hincque

$$\frac{\partial u}{u} = \frac{(s-1)\partial s}{\sqrt{(m s s + n)} - s s + s},$$

ex qua fit

$$\frac{\partial x}{x} = \frac{\partial u}{(s-1)u} = \frac{\partial s}{\sqrt{(m s s + n)} - s s + s},$$

unde tam $u = \frac{y}{x}$ quam x per eandem variabilem s determinatur.

Exemplum 3.

804. *Sumto elemento ∂x constante, si proponatur haec aequatio $n x^3 \partial \partial y = (y \partial x - x \partial y)^2$, ejus integrale invenire.*

Posito $\partial y = p \partial x$ et $\partial p = q \partial x$, erit $n x^3 q = (y - p x)^2 = n x x v$ ob $q = \frac{v}{x}$. Si jam statuatur $y = u x$, fiet

$$n v = (u - p)^2 \text{ et } \frac{\partial x}{x} = \frac{\partial u}{p - u} = \frac{\partial p}{v} = \frac{n \partial p}{(p-u)^2},$$

unde habetur $n \partial p = p \partial u - u \partial u$: quae facto $p - u = s$, abit in $n \partial u + n \partial s = s \partial u$ seu $\partial u = \frac{n \partial s}{s - n}$, hinc $u = n\, l \frac{s-n}{\alpha}$. Tum vero ob $p - u = s$, erit

$$\frac{\partial x}{x} = \frac{\partial u}{s} = \frac{n \partial s}{s(s-n)} \text{ et } l x = l \frac{s-n}{\beta s}, \text{ ideoque}$$

$$x = \frac{s-n}{\beta s}, \text{ at } y = n x\, l \frac{s-n}{\alpha}.$$

Cum ergo sit

$s = \frac{n}{1-\beta x}$, erit $y = nxl\frac{n\beta x}{\alpha(1-\beta x)}$.

Corollarium.

805. Aequatio $nx^3q = (y-px)^2$ facilius resolvitur, ponendo $y-px=z$, unde fit $-x\partial p = \partial z$; quare ob $q\partial x = \partial p$, erit

$$nx^3\partial p = zz\partial x = -nxx\partial z,$$

ideoque

$$\frac{1}{x} = \frac{1}{a} - \frac{n}{z} \text{ seu } \frac{x-a}{ax} = \frac{n}{y-px}, \text{ ergo}$$
$$y - px = \frac{nax}{x-a} = \frac{y\partial x - x\partial y}{\partial x}.$$

Quare

$$\frac{y\partial x - x\partial y}{xx} = \frac{na\partial x}{x(x-a)} \text{ et}$$
$$\frac{y}{x} = nl\frac{x}{x-a} + C \text{ ut ante.}$$

Exemplum 4.

806. *Sumto ∂x constante, si proponatur haec aequatio*
$$(\partial x^2 + \partial y^2)\sqrt{(\partial x^2 + \partial y^2)} = n\partial x\partial\partial y\sqrt{(xx+yy)},$$
integrale invenire.

Posito $\partial y = p\partial x$ et $\partial p = q\partial x$, aequatio proposita hanc induit formam

$$(1+pp)\sqrt{(1+pp)} = nq\sqrt{(xx+yy)},$$

quae facto $y = ux$ et $q = \frac{v}{x}$ transit in hanc

$$(1+pp)\sqrt{(1+pp)} = nv\sqrt{(1+uu)}.$$

Cum igitur sit $\frac{\partial x}{x} = \frac{\partial u}{p-u} = \frac{\partial p}{v}$, ob

$$v = \frac{(1+pp)\sqrt{(1+pp)}}{n\sqrt{(1+uu)}} \text{ erit}$$

$$(1+pp)^{\frac{3}{2}}\partial u = n(p-u)\partial p\sqrt{(1+uu)}$$

cujus resolutio non sponte patet. Calculum autem ad angulos revocando, sit $p = \text{tang.}\,\Phi$ et $u = \text{tang.}\,\omega$, erit

$$\partial p = \frac{\partial \Phi}{\cos. \Phi^2},\ \partial u = \frac{\partial \omega}{\cos. \omega^2},$$

$$\sqrt{(1 + pp)} = \frac{1}{\cos. \Phi},\ \sqrt{(1 + uu)} = \frac{1}{\cos. \omega},$$ et

$$p - u = \frac{\sin. (\Phi - \omega)}{\cos. \Phi \cos. \omega},$$ hincque

$$\frac{1}{\cos. \Phi^3} \cdot \frac{\partial \omega}{\cos. \omega^2} = \frac{n \sin. (\Phi - \omega)}{\cos. \Phi \cos. \omega} \cdot \frac{1}{\cos. \omega} \cdot \frac{\partial \Phi}{\cos. \Phi^2}$$ sive

$$\partial \omega = n \partial \Phi \sin. (\Phi - \omega) = \partial \Phi - (\partial \Phi - \partial \omega)$$ ergo

$$\partial \Phi = \frac{\partial \Phi - \partial \omega}{1 - n \sin. (\Phi - \omega)}.$$

Posito ergo $\Phi - \omega = \psi$, erit

$$\Phi = \int \frac{\partial \psi}{1 - n \sin. \psi} \text{ et } \omega = \int \frac{\partial \psi}{1 - n \sin. \psi} - \psi;$$

hinc ob $p = \text{tang.} \Phi$, et $u = \text{tang.} \omega$, obtinetur

$$\frac{\partial x}{x} = \frac{\partial u \cos. \Phi \cos. \omega}{\sin. \psi} = \frac{\partial \omega \cos. \Phi}{\sin. \psi \cos. \omega} = \frac{n \partial \psi \cos. \Phi}{\cos. \omega (1 - n \sin. \psi)}.$$

Casu $n = 1$, fit

$$\partial \Phi = \frac{\partial \psi}{1 - \sin. \psi} = \frac{\partial \psi (1 + \sin. \psi)}{\cos. \psi^2},$$

$$\Phi = \text{tang.} \psi + \frac{1}{\cos. \psi} = \frac{1 + \sin. \psi}{\cos. \psi} + \alpha,$$

$$\omega = \frac{1 + \sin. \psi}{\cos. \psi} + \alpha - \psi,$$ et

$$\frac{\partial x}{x} = \frac{\partial \Phi \cos. \Phi}{\cos. \omega} = \frac{\partial \Phi \cos. \Phi}{\cos. \Phi \cos \psi + \sin. \Phi \sin. \psi}.$$

Cum autem sit $\Phi - \alpha = \sqrt{\frac{1 + \sin. \psi}{1 - \sin. \psi}}$, erit

$$\sin \psi = \frac{(\Phi - \alpha)^2 - 1}{(\Phi - \alpha)^2 + 1},\ \cos. \psi = \frac{2(\Phi - \alpha)}{(\Phi - \alpha)^2 + 1}.$$ Ergo

$$\frac{\partial x}{x} = \frac{\partial \Phi \cos. \Phi [(\Phi - \alpha)^2 + 1]}{2 (\Phi - \alpha) \cos. \Phi + (\Phi - \alpha)^2 \sin. \Phi - \sin. \Phi}.$$

Corollarium 3.

807. Si casu $n = 1$ sumatur constans α infinita, erit $\sin \psi = 1$, hinc $\psi = 90^\circ$ et $\omega = \Phi - 90^\circ$, tum vero $\frac{\partial x}{x} = \frac{\partial \Phi \cos. \Phi}{\sin. \Phi}$, ideoque $x = a \sin. \Phi$, et $u = -\cot. \Phi$, ergo $y = -a \cos. \Phi$, et $xx + yy = aa$.

Corollarium 2.

808. Eodem autem casu $n=1$, quo constans α non sumitur infinita, fractionis, cui $\frac{\partial x}{x}$ aequatur, numerator commode est differentiale denominatoris; unde fit

$$x=a\,[(\Phi-\alpha)^2 \sin.\Phi-\sin.\Phi+2(\Phi-\alpha)\cos.\Phi].$$

Tum vero est

$$\omega=\Phi-\text{Ang. tang.}\,\frac{(\Phi-\alpha)^2-1}{2(\Phi-\alpha)},$$

ideoque

$$u=\frac{y}{x}=\text{tang.}\,\omega=\frac{\text{tang.}\,\Phi-\frac{(\Phi-\alpha)^2+1}{2(\Phi-\alpha)}}{1+\frac{(\Phi-\alpha)^2-1}{2(\Phi-\alpha)}\,\text{tang.}\,\Phi},\ \text{seu}$$

$$\frac{y}{x}=\frac{2(\Phi-\alpha)\sin.\Phi-(\Phi-\alpha)^2\cos.\Phi+\cos.\Phi}{(\Phi-\alpha)^2\sin.\Phi-\sin.\Phi+2(\Phi-\alpha)\cos.\Phi};$$

consequenter

$$y=-a[(\Phi-\alpha)^2\cos.\Phi-\cos.\Phi-2(\Phi-\alpha)\sin.\Phi]\ \text{et}$$

$$\sqrt{(xx+yy)}=[(\Phi-\alpha)^2+1].$$

Scholion 1.

809. In genere etiam integrationem absolvere licet. Cum enim sit

$$\partial\Phi=\frac{\partial\psi}{1-n\sin.\psi}\ \text{et}\ \frac{\partial x}{x}=\frac{n\partial\Phi\cos.\Phi}{\cos.\omega},\ \text{erit}$$

$$\Phi+\alpha=\frac{1}{\sqrt{(1-nn)}}\,\text{Ang. cos.}\,\frac{n-\sin.\psi}{1-n\sin.\psi};$$

unde posito

$$(\Phi+\alpha)\sqrt{(1-nn)}=\theta,\ \text{erit}\ \cos.\theta=\frac{n-\sin.\psi}{1-n\sin.\psi};$$

hincque

$$\sin.\psi=\frac{n-\cos.\theta}{1-n\cos.\theta}\ \text{et}\ \cos.\psi=\frac{\sin.\theta\sqrt{(1-nn)}}{1-n\cos.\theta}.$$

At ob $\omega=\Phi-\psi$, habebitur

$$\frac{\partial x}{x}=\frac{n\partial\Phi\cos.\Phi(1-n\cos.\theta)}{\cos.\Phi\sin.\theta\sqrt{(1-nn)}+\sin.\Phi(n-\cos.\theta)}.$$

Jam cum sit $\partial\theta = \partial\Phi\sqrt{(1-nn)}$, differentiale hujus denominatoris est

$$-\partial\Phi \sin.\Phi \sin.\theta\sqrt{(1-nn)} + \partial\Phi\cos.\Phi\cos.\theta\,(1-nn)$$
$$+n\partial\Phi\cos.\Phi - \partial\Phi\cos.\Phi\cos.\theta + \partial\Phi\sin.\Phi\sin.\theta\sqrt{(1-nn)},$$

quod redit ad $n\partial\Phi\cos.\Phi(1-n\cos.\theta)$ ipsum scilicet numeratorem. Ita ut sit

$$x = a\,[\cos.\Phi\sin.\theta\sqrt{(1-nn)} + \sin.\Phi(n-\cos.\theta)]$$

seu $x = a\cos.\omega(1-n\cos.\theta)$, ideoque

$$y = ux = a\sin.\omega\,(1-n\cos\theta).$$

Assumto ergo angulo θ, quaeratur angulus ψ, ut sit

$$\sin.\psi = \frac{n-\cos.\theta}{1-n\cos.\theta} \text{ et } \cos.\psi = \frac{\sin.\theta}{1-n\cos.\theta}\sqrt{(1-nn)},$$

tum vero fiat

$$\omega = \frac{\theta}{\sqrt{(1-nn)}} - \alpha - \psi,$$

eritque integrale completum

$$x = a\,(1-n\cos.\theta)\cos.\omega \text{ et } y = a\,(1-n\cos.\theta)\sin.\omega.$$

Scholion 2.

810. At si numerus n sit unitate major, haec integratio fit ima inaria, quod incommodum ut tollatur, notandum est, aequationis $\partial\Phi = \frac{\partial\psi}{1-n\sin.\psi}$ integrale esse

$$\Phi + \alpha = \frac{1}{\sqrt{(nn-1)}}\, l\frac{\sqrt{(n-1)(1+\sin.\psi)} + \sqrt{(n+1)(1-\sin.\psi)}}{\sqrt{(n-1)(1+\sin.\psi)} - \sqrt{(n-1)(1-\sin.\psi)}}.$$

Quare si ponatur $(\Phi+\alpha)\sqrt{(nn-1)} = \theta$, ut sit

$$\partial\theta = \partial\Phi\sqrt{(nn-1)} \text{ et } \omega = \Phi - \psi = \frac{1}{\sqrt{(nn-1)}} - \alpha - \psi,$$

erit

$$\frac{e^\theta+1}{e^\theta-1} = \frac{\sqrt{(n-1)(1+\sin.\psi)}}{\sqrt{(n+1)(1-\sin.\psi)}} = \frac{(n-1)(1+\sin.\psi)}{\cos.\psi\sqrt{(nn-1)}};$$

unde reperitur

$$\sin.\psi = \frac{e^\theta + 2n + e^{-\theta}}{ne^\theta + 2 + ne^{-\theta}} \text{ et } \cos.\psi = \frac{(e^\theta - e^{-\theta})\sqrt{(nn-1)}}{ne^\theta + 2 + ne^{-\theta}},$$

ita ut ex angulo θ definiantur anguli ψ, Φ et ω. Cum jam sit

$$\frac{\partial x}{x} = \frac{n\partial\Phi\cos.\Phi}{\cos.\omega} = \frac{n\partial\Phi\cos.\Phi}{\cos.\Phi\cos.\psi + \sin.\Phi\sin.\psi}, \text{ erit}$$

$$\frac{\partial x}{x} = \frac{n\partial\Phi\cos.\Phi(ne^\theta + 2 + ne^{-\theta})}{\cos.\Phi(e^\theta - e^{-\theta})\sqrt{(nn-1)} + \sin.\Phi(e^\theta + 2n + e^{-\theta})},$$

ubi iterum commode evenit, ut numerator sit ipsum differentiale denominatoris, quemadmodum differentiationem instituenti mox patebit. Hinc ergo erit

$$x = a[\cos.\Phi(e^\theta - e^{-\theta})\sqrt{(nn-1)} + \sin.\Phi(e^\theta + 2n + e^{-\theta})] \text{ seu}$$

$$x = a\cos.\omega(ne^\theta + 2 + ne^{-\theta}), \text{ et ob } u = \frac{y}{x} = \text{tang.}\,\omega \text{ fit}$$

$$y = a\sin.\omega(ne^\theta + 2 + ne^{-\theta}).$$

Quocirca ex angulo θ primo quaeratur angulus ψ, ut sit

$$\sin.\psi = \frac{e^\theta + 2n + e^{-\theta}}{ne^\theta + 2 + ne^{-\theta}} \text{ et } \cos.\psi = \frac{(e^\theta - e^{-\theta})\sqrt{(nn-1)}}{ne^\theta + 2 + ne^{-\theta}},$$

quo invento capiatur angulus $\omega = \frac{\theta}{\sqrt{(nn-1)}} - \alpha - \psi$, ac formulae illae pro x et y inventae dabunt integrale completum ob duas constantes a et α introductas.

Scholion 3.

811. Cum hic praecipua pars integrationis fortuito successisse videatur, operae praetium erit in ejus causam inquirere, num forte ratio integrandi clarius perspici queat. Cum igitur sit

$$\Phi = \psi + \omega \text{ et } \partial\Phi = \frac{\partial\psi}{1 - n\sin.\psi},$$

hincque

$$\partial\omega = \frac{n\partial\psi\sin.\psi}{1 - n\sin.\psi} = n\partial\Phi\sin.\psi,$$

aequatio

$$\frac{\partial x}{x} = \frac{n \partial \Phi \cos.\Phi}{\cos.\omega}, \text{ ob } \cos.\Phi = \cos.\psi \cos.\omega - \sin.\psi \sin.\omega,$$

in hanc resolvitur

$$\frac{\partial x}{x} = n \partial \Phi \cos.\psi - \frac{n \partial \Phi \sin.\psi \sin.\omega}{\cos.\omega},$$

quae ob

$$\partial \Phi = \frac{\partial \psi}{1 - n \sin.\psi} \text{ et } n \partial \Phi \sin.\psi = \partial \omega,$$

induit hanc formam integrabilem

$$\frac{\partial x}{x} = \frac{n \partial \psi \cos.\psi}{1 - n \sin.\psi} - \frac{\partial \omega \sin.\omega}{\cos.\omega},$$

ex qua elicitur

$$x = \frac{a \cos.\omega}{1 - n \sin.\psi} \text{ et } y = \frac{a \sin.\omega}{1 - n \sin.\psi}, \text{ ob } y = ux = x \tang.\omega.$$

En ergo in genere aequationis nostrae hanc integrationem. Anguli ω et ψ hanc inter se teneant relationem, ut sit

$$\partial \omega = \frac{n \partial \psi \sin.\psi}{1 - n \sin.\psi},$$

tum vero erit

$$x = \frac{a \cos.\omega}{1 - n \sin.\psi} \text{ et } y = \frac{a \sin.\omega}{1 - n \sin.\psi}.$$

Quodsi ergo ponamus $\sqrt{(xx + yy)} = z$, ut sit

$x = z \cos\omega$ et $y = z \sin.\omega$, erit

$z = \frac{a}{1 - n \sin.\psi}$ et $\sin.\psi = \frac{z - a}{nz}$, hinc

$\partial \omega = \frac{(z - a) \partial \psi}{a}$. At fit

$\partial \psi = \frac{a \partial z}{z \sqrt{[nnzz - (z - a)^2]}}$ ergo

$$\partial \omega = \frac{(z - a) \partial z}{z \sqrt{[nnzz - (z - a)^2]}}$$

unde angulus ω per z definitur. Ad irrationalitatem tollendam, si ponamus

$$\sqrt{[nnzz - (z - a)^2]} = s(nz + z - a), \text{ fit}$$

**

$$z = \frac{a(ss+1)}{(n+1)ss-n+1} \text{ et } \partial\omega = \frac{2n\partial s(ss-1)}{(ss+1)[(n+1)ss-(n-1)]}, \text{ seu}$$

$$\partial\omega = \frac{2\partial s}{ss+1} - \frac{2\partial s}{(n+1)ss-n+1}$$

quae integratio est manifesta.

Problema 99.

812. Si aequatio differentio-differentialis tum demum fiat homogenea, si alteri variabili y tribuantur n dimensiones, ejus integrationem ad aequationem differentialem primi gradus reducere.

Solutio.

Posito $\partial y = p\partial x$ et $\partial p = q\partial x$, ut oriatur aequatio inter quaternas quantitates finitas x, y, p et q, quae quomodo ratione homogeneitatis futura sit comparata, videamus. Primo ergo cum pro x unam dimensionem numerando, variabilis y habeat n dimensiones, quantitati $p = \frac{\partial y}{\partial x}$ tribuendae sunt $n-1$ dimensiones, quantitati $q = \frac{\partial p}{\partial x}$ vero $n-2$ dimensiones. Quocirca ponamus

$$y = x^n u,\ p = x^{n-1} t, \text{ et } q = x^{n-2} v,$$

et ob $\partial y = p\partial x$ et $\partial p = q\partial x$, habebimus

$$x\partial u + nu\partial x = t\partial x \text{ et } x\partial t + (n-1)t\partial x = v\partial x,$$

unde colligimus

$$\frac{\partial x}{x} = \frac{\partial u}{t-nu} = \frac{\partial t}{v-(n-1)t},$$

ideoque

$$\partial u[v-(n-1)t] = \partial t(t-nu).$$

At factis superioribus substitutionibus in aequatione inter x, y, p et q, per hypothesin variabilis x ex calculo extruditur, ita ut prodeat aequatio inter tres tantum variabiles u, t et v, ex qua litteram v per binas t et u definire licebit. Quo valore substituto habebitur aequatio differentialis primi gradus inter binas variabiles u

et t, ex qua t per u determinari queat; ope aequationis $\frac{\partial x}{x} = \frac{\partial u}{t - nu}$ definietur x per u, hincque ob $u = \frac{y}{x^n}$ obtinebitur aequatio integralis inter x et y, eaque ob duplicem integrationem completa.

Corollarium 1.

813. Aequationum ergo inter x, y, p et q hoc modo tractabilium hoc est criterium, ut posito $y = x^n u$, $p = x^{n-1} t$ et $q = x^{n-2} v$, exponens n ejusmodi determinationem patiatur, ut variabilis x prorsus ex calculo per divisionem egrediatur.

Corollarium 2.

814. Si sit $n = 0$, aequatio ita est comparata, ut tribuendo ipsi y ejusque differentialibus nullam dimensionem fiat homogenea. Hoc scilicet casu sola variabilis x cum suis differentialibus dimensiones constituere censetur.

Corollarium 3.

815. Contra vero si dimensiones ex sola variabili y aestimentur, ita ut ea cum suis differentialibus ∂y et $\partial \partial y$ ubique eundem dimensionum numerum constituat, exponens n fiet infinitus.

Scholion.

816. Si sola variabilis x cum suis differentialibus ubique eundem dimensionum numerum complet, ob $n = 0$ fit $u = y$, et in aequatione inter x, y, p et q statui conveniet $p = \frac{t}{x}$ et $q = \frac{v}{xx}$ quo facto variabilis x ex calculo deturbabitur, prodibitque aequatio inter y, t et v, cujus ope aequatio differentialis $\partial y (v + t) = t \partial t$ ad duas tantum variabiles reducetur, qua resoluta erit $\frac{\partial x}{x} = \frac{\partial y}{t}$, ubi cum t detur per y, integratio nullam habet difficultatem. Ve-

rum altero casu, quo variabilis y sola cum suis differentialibus pares ubique dimensiones habet, ideoque exponens n infinitus capi deberet, resolutio alio modo institui debet, quem mox docebimus, nisi forte permutando variabiles x et y casum ad praecedentem reducere lubuerit.

Exemplum 1.

817. *Sumto elemento ∂x constante, si proponatur haec aequatio $xx\partial\partial y = \alpha y \partial x^2 + \beta x \partial x \partial y$, ejus integrale invenire.*

Perpendatur hic ista conditio, qua sola variabilis x cum suo differentiali ∂x ubique duas constituit dimensiones, eritque $n = 0$. Cum ergo posito $\partial y = p \partial x$ et $\partial p = q \partial x$, habeamus $q x x = \alpha y + \beta p x$, statuamus $p = \frac{t}{x}$ et $q = \frac{v}{xx}$, fietque $v = \alpha y + \beta t$, unde adipiscimur istam aequationem differentialem

$$\alpha y \partial y + (\beta + 1) t \partial y = t \partial t,$$

ob cujus homogeneitatem faciamus $t = y z$, eritque

$$\alpha \partial y + (\beta + 1) z \partial y = y z \partial z + z z \partial y \text{ seu}$$
$$\frac{\partial y}{y} = \frac{z \partial z}{\alpha + (\beta + 1) z - z z}.$$

Sit $\alpha + (\beta + 1) z - z z = (f + z)(g - z)$

ut sit

$$\alpha = f g \text{ et } \beta + 1 = g - f,$$

reperieturque

$$\frac{\partial y}{y} = \frac{\frac{-f}{f+g} \partial z}{f + z} + \frac{\frac{g}{f+g} \partial z}{g - z}:$$

unde colligitur integrando

$$l y = C - \frac{f}{f+g} l (f + z) - \frac{g}{f+g} l (g - z), \text{ seu}$$

$$y (f + z)^{\frac{f}{f+g}} (g - z)^{\frac{g}{f+g}} = a.$$

Tum vero est

$$\frac{\partial x}{x} = \frac{\partial y}{y z} = \frac{\partial z}{(f+z)(g-z)}, \text{ seu}$$

$$\frac{\partial x}{x} = \frac{1}{f+g} \cdot \frac{\partial z}{f+z} + \frac{1}{f+g} \cdot \frac{\partial z}{g-z}, \text{ hinc}$$

$$x = b \left(\frac{f+z}{g-z}\right)^{\frac{1}{f+g}}, \text{ seu } \frac{f+z}{g-z} = \left(\frac{x}{b}\right)^{f+g}.$$

Unde cum sit $z = \dfrac{g\, x^{f+g} - f\, b^{f+g}}{b^{f+g} + x^{f+g}}$, erit hoc valore ibi substituto

$$(f+g)\, b^g\, x^f\, y = a\,(b^{f+g} + x^{f+g}),$$

seu posito $\frac{a}{f+g} = c$

$$y = c\left(\frac{b^f}{x^f} + \frac{x^g}{b^g}\right):$$

est vero $g - f = \beta + 1$ et $g + f = \sqrt{((\beta+1)^2 + 4\alpha)}$.

Corollarium.

818. Quoniam in aequatione proposita etiam ambae variabiles x et y simul ubique totidem dimensiones habent, eam etiam secundum praecepta praecedentis problematis tractare licet.

Exemplum 2.

819. *Posito ∂x constante, si aequatio differentio-differentialis duobus tantum terminis constet, ut sit hujusmodi*

$$\partial \partial y = c\, x^\alpha\, y^\beta\, \partial x^{2-\gamma}\, \partial y^\gamma,$$

ejus integrale investigare.

Posito $\partial y = p\, \partial x$ et $\partial p = q\, \partial x$, habebitur haec forma $q = c\, x^\alpha\, y^\beta\, p^\gamma$, ubi exponentem n ita definire licet, ut posito $y = x^n u$, $p = x^{n-1} t$ et $q = x^{n-2} v$, variabilis x per divisionem tolli possit, capi enim oportet

$$\alpha+\beta+\gamma(n-1)-n+2=0, \quad \text{seu } n=\frac{-\alpha+\gamma-2}{\beta+\gamma-1};$$

tumque erit $v=cu^{\beta}t^{\gamma}$. Aequatio ergo differentialis primi gradus resolvenda erit

$$cu^{\beta}t^{\gamma}\partial u-(n-1)t\partial u=t\partial t-nu\partial t,$$

ex qua cum variabilis t per u fuerit determinata, integrari oportet hanc formulam $\frac{\partial x}{x}=\frac{\partial u}{t-nu}$, quo facto ob $u=\frac{y}{x^n}$, obtinebitur aequatio integralis quaesita inter x et y.

Casus tantum $\beta+\gamma=1$, quo n fit infinitus, peculiarem postulat tractationem infra exponendam, nisi forte simul sit $\gamma=\alpha+2$, tum enim exponens n prorsus arbitrio nostro relinquitur, at aequatio erit homogenea.

Exemplum 3.

820. *Sumto elemento ∂x constante, si proponatur haec aequatio*

$$x^4\partial\partial y=x^3\partial x\partial y+2xy\partial x\partial y-4yy\partial x^2,$$

ejus integrale invenire.

Hic evidens est, si ipsi y ejusque differentialibus ∂y et $\partial\partial y$ binae dimensiones, ipsi x vero et ∂x singulae tribuantur, in omnibus terminis obtineri sex dimensiones. Quare cum posito $\partial y=p\partial x$ et $\partial p=q\partial x$, habeamus hanc aequationem

$x^4q=x^3p+2xyp-4yy$, faciamus

$y=x^2u, p=xt$ et $q=v$, prodibitque

$v=t+2ut-4uu.$

At ob $n=2$, aequatio differentialis nostra erit

$$\partial u(v-t)=\partial t(1-2u),$$

quae abit in

$$2u\partial u(t-2u)=\partial t(t-2u),$$

unde deducimus vel $t=2u$, vel $t=uu+c$, quos binos casus seorsim evolvamus.

1) Si $t=2u$, ob $\frac{\partial x}{x}=\frac{\partial u}{t-2u}$ fit $\partial u=0$, ideoque $u=C$, ac propterea $y=Cxx$, quod est integrale particulare, aequationi propositae utique satisfaciens.

2) Sit $t=uu+c$, erit $\frac{\partial x}{x}=\frac{\partial u}{uu-2u+c}$, ubi tres casus sunt considerandi:

Primo si constans $c=1$, erit

$$l\frac{x}{a}=\frac{1}{1-u}=\frac{xx}{xx-y}, \text{ seu } xx=(xx-y)\,l\frac{x}{a}.$$

Secundo si constans $c=1-ff$, erit $\frac{\partial x}{x}=\frac{\partial u}{(u-1)^2-ff}$, hincque

$$lx=\frac{-1}{2f}\,l\frac{f+u-1}{f-u+1}+C,$$

ergo ob $u=\frac{y}{xx}$ erit

$$x=a\left(\frac{(f+1)xx-y}{(f-1)xx+y}\right)^{\frac{1}{2f}}.$$

Tertio si constans $c=1+ff$, ideoque $\frac{\partial x}{x}=\frac{\partial u}{(u-1)^2+ff}$, quae integrata dat

$$l\frac{x}{a}=\frac{1}{f}\,\text{Ang. tang.}\frac{u-1}{f}, \text{ seu } \frac{u-1}{f}=\frac{y-xx}{fxx}=\text{tang.}\,f\,l\frac{x}{a}.$$

Pro ratione ergo constantis arbitrariae c integratio vel algebraice succedit, vel a logarithmis, vel ab angulis pendet, unde forma generali exprimi nequit.

Scholion.

821. Integrale autem particulare primo inventum $y=Cxx$ in nulla harum formarum, quibus integrale completum constituitur,

contineri deprehenditur: nihilo vero minus satisfacit aequationi differentio-differentiali propositae. Hoc ergo exemplo magis illustrantur ea, quae supra circa hoc paradoxon sumus commentati, quod interdum aequationi differentiali satisfaciat aequatio finita, quae in integrali completo minime contineatur. Videmus igitur hoc idem paradoxon etiam in aequationibus differentio-differentialibus locum habere. Utrum autem illa aequatio $y = Cxx$ inter integralia sit admittenda, alia est quaestio, quae nondum penitus videtur confecta; hic quidem ipsa aequatio proposita quasi factores habens est censenda, ex quorum altero illa aequatio $y = Cxx$ nascatur, verum multum abest, ut in hac explicatione acquiescere queamus. Quin potius ipsa illa quaestio, sive geometrica fuerit sive alius disciplinae, cujus solutio ad hujusmodi aequationem perduxerit, accurate perpendi debere videtur; ubi plerumque haud difficulter indicare potest, utrum quicquid aequationi differentiali satisfaciat, id etiam ipsi quaestioni conveniat nec ne? Veluti si descensus gravis ex altitudine $= a$ labentis definiri debeat, et altitudo qua jam a terra distat sit x, erit ibi celeritas ut $\sqrt{(a - x)}$, et elementum temporis $\partial t = \frac{-\partial x}{\sqrt{(a - x)}}$. Hic quidem evidens est isti aequationi differentiali satisfieri ponendo $x = a$, ita ut tempus t maneat indefinitum, quod tamen quaestioni neutiquam convenit, quae nonisi vero integrali $t = 2\sqrt{(a - x)}$ resolvitur.

Problema 100.

822. Si in aequatione differentio-differentiali variabilis y cum suis differentialibus ∂y et $\partial \partial y$ ubique eundem dimensionum numerum adimpleat, ejus integrationem ad aequationem differentialem primi gradus reducere.

Solutio.

Posito $\partial y = p \partial x$ et $\partial p = q \partial x$, aequatio ita erit comparata, ut in ea ternae variabiles y, p, q, ubique eundem dimensio-

num numerum obtineant, altera variabili x in computum dimensionum prorsus non ingrediente. Quare si statuatur $p = uy$ et $q = vy$, in omnibus terminis inerit eadem ipsius y potestas, qua per divisionem sublata habebitur aequatio inter ternas tantum variabiles x, u et v, ex qua unam per binas reliquas definire licebit, ita ut v aequetur functioni cuipiam ipsarum x et u. Jam ob $p = uy$ erit $\partial y = uy\partial x$, et ob $\partial p = q\partial x$ fiet $u\partial y + y\partial u = vy\partial x$, unde sequitur

$$\frac{\partial y}{y} = u\partial x \text{ et } \frac{\partial y}{y} = \frac{v\partial x - \partial u}{u},$$

ideoque $\partial u + uu\partial x = v\partial x$, quae aequatio differentialis duas tantum variabiles x et u complectitur. Quam ergo si integrare liceat, ut relatio inter x et u inde innotescat, superest, ut formulae $u\partial x$ integrale investigetur, quo invento erit $ly = \int u\partial x$, sicque aequatio orietur integralis inter x et y, quae ob duplicem integrationem peractam duas constantes arbitrarias involvet, ideoque integrale completum exhibebit.

Corollarium 1.

823. Hujusmodi ergo aequationum integratio reducitur ad hujusmodi aequationem differentialem $\partial u + uu\partial x = v\partial x$, cujus resolutio si succedat, simul illarum integratio habetur, cum formulae $u\partial x$ integratio difficultate careat.

Corollarium 2.

824. Cum sit $\frac{\partial y}{y} = u\partial x$, erit $y = e^{\int u\partial x}$, qua substitutione aequatio differentio-differentialis proposita statim reducitur ad aequationem differentialem primi gradus, erit enim

$$\frac{\partial y}{\partial x} = p = e^{\int u\partial x}u, \text{ et } \frac{\partial p}{\partial x} = q = \frac{e^{\int u\partial x}(\partial u + uu\partial x)}{\partial x}.$$

ac tum formula exponentialis sponte ex aequatione egreditur.

Corollarium 3.

825. Vicissim etiam proposita aequatione differentiali primi gradus $\partial u + uu\partial x = v\partial x$, in qua v sit functio quaecunque ipsarum x et u, ea posito $u = \frac{\partial y}{y\partial x}$, in ejusmodi aequationem differentio-differentialem transformatur, in qua variabilis y cum suis differentialibus ∂y et $\partial\partial y$ ubique eundem dimensionum numerum constituat.

Scholion 1.

826. Haec reductio aequationum differentialium primi gradus ad gradum secundum legibus analyseos adversari videtur, interim tamen subinde usu non caret; quodsi enim alia methodo hujusmodi aequationes differentio-differentiales tractare liceat, dum earum integralia vel per series exhibentur vel finite, simul integralia aequationum differentialium primi gradus innotescunt, quorum ratio plerumque aliunde vix perspicitur. In sequentibus autem videbimus, ejusmodi aequationes differentio-differentiales in quibus variabilis altera y unam dimensionem non superat, per series commode integrari posse, atque adeo interdum has series abrumpi, ita ut integrale finita expressione exhibeatur. Caeterum proposita hujusmodi aequatione differentiali primi gradus $\partial u + uu\partial x = v\partial x$, substitutio $u = \frac{\partial y}{y\partial x}$ eo magis est notatu digna, quod sumto elemento ∂x constante fiat

$$\partial u = \frac{\partial\partial y}{y\partial x} - \frac{\partial y^2}{yy\partial x}, \text{ ideoque}$$

$$\partial u + uu\partial x = \frac{\partial\partial y}{y\partial x},$$

ita ut duo termini hoc modo in unum coalescant.

Scholion 2.

827. Casus hic imprimis notasse juvabit, quibus aequatio $\partial u + uu\partial x = v\partial x$ integrationem admittit. Hunc in finem sit

$\partial u = V \partial x$ forma generalis aequationum resolubilium, et V certa functio ipsarum x et u, ac manifestum est, si fuerit $v = u u + V$, integrationem succedere. Primum ergo hoc eveniet si sit $V = \frac{X}{U}$, denotante X functionem ipsius x, et U ipsius u. Secundo si V sit functio homogenea nullius dimensionis ipsarum x et u. Tertio si denotantibus X et Ξ functiones quascunque ipsius x, fuerit $V = X u + \Xi u^n$. Quarto si denotantibus P et Q functiones quascunque ipsius u, fuerit $V = \frac{1}{P x + Q x^n}$. Similique modo ex aliis formis integrabilibus alii casus concludentur.

Exemplum 1.

828. *Sumto elemento ∂x constante, si proponatur haec aequatio*

$$\alpha y \partial\partial y + \beta \partial y^2 = \frac{y \partial x \partial y}{\sqrt{(aa + xx)}};$$

ejus integrale invenire.

Posito $\partial y = p \partial x$ et $\partial p = q \partial x$, prodit

$$\alpha y q + \beta p p = \frac{y p}{\sqrt{(aa + xx)}},$$

quae facto $p = u y$ et $p = v y$, abit in hanc

$$\alpha v + \beta u u = \frac{u}{\sqrt{(aa + xx)}}, \text{ seu } v = \frac{u}{\alpha \sqrt{(aa + xx)}} - \frac{\beta u u}{\alpha},$$

unde hanc aequationem resolvi oportet

$$\partial u + u u \partial x = \frac{u \partial x}{\alpha \sqrt{(aa + xx)}} - \frac{\beta u u \partial x}{\alpha}.$$

Statuatur $u = \frac{1}{s}$, fietque

$$+ \partial s + \frac{s \partial x}{\alpha \sqrt{(aa + xx)}} = \left(1 + \frac{\beta}{\alpha}\right) \partial x,$$

quae per $[x + \sqrt{(aa + xx)}]^{\frac{1}{\alpha}}$ multiplicata et integrata dat

$$s[x + \sqrt{(aa + xx)}]^{\frac{1}{\alpha}} = \left(1 + \frac{\beta}{\alpha}\right) \int \partial x [x + \sqrt{(aa + xx)}]^{\frac{1}{\alpha}}.$$

Fiat $x+\sqrt{(aa+xx)}=t^{\alpha}$, erit

$$aa=t^{2\alpha}-2t^{\alpha}x, \text{ hinc}$$

$$x=\frac{t^{2\alpha}-aa}{2t^{\alpha}}=\tfrac{1}{2}t^{\alpha}-\tfrac{1}{2}aat^{-\alpha} \text{ et}$$

$$\partial x=\tfrac{\alpha}{2}\partial t(t^{\alpha-1}+aat^{-\alpha-1});$$

ita ut sit

$$st=(1+\tfrac{\beta}{\alpha})\int\tfrac{\alpha}{2}\partial t\,(t^{\alpha}+aat^{-\alpha}) \text{ seu}$$

$$st=C+\frac{\alpha+\beta}{\alpha}\left(\frac{t^{\alpha+1}}{\alpha+1}+\frac{aat^{1-\alpha}}{1-\alpha}\right).$$

Porro est $\frac{\partial y}{y}=u\partial x=\frac{\partial x}{s}$; at ex aequatione differentiali est

$$(1+\tfrac{\beta}{\alpha})\frac{\partial x}{s}=\frac{\partial s}{s}+\frac{\partial x}{\alpha\sqrt{(aa+xx)}},$$

hincque

$$(1+\tfrac{\beta}{\alpha})\,ly=ls+\tfrac{1}{\alpha}\,l[x+\sqrt{(aa+xx)}]=lst+D$$

ergo $y=B(st)^{\frac{\alpha}{\alpha+\beta}}$. Quare sumto $C=\frac{\alpha+\beta}{2}A$, habebitur

$$y=B\left(A+\frac{t^{\alpha+1}}{\alpha+1}+\frac{aat^{1-\alpha}}{1-\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}$$

existente

$$x=\tfrac{1}{2}(t^{\alpha}-aat^{-\alpha}), \text{ vel } t=[x+\sqrt{(aa+xx)}]^{\frac{1}{\alpha}},$$

ita ut aequatio inter x et y sit

$$Cy^{\frac{\alpha+\beta}{\alpha}}=A+\tfrac{1}{\alpha+1}[x+\sqrt{(aa+xx)}]^{\frac{\alpha+1}{\alpha}}$$
$$+\tfrac{aa}{1-\alpha}[x+\sqrt{(aa+xx)}]^{\frac{1-\alpha}{\alpha}}.$$

Scholion.

829. Hoc idem exemplum ita est comparatum, ut alia ratione facillime resolvi possit; aequatio enim

$$\alpha yq+\beta pp=\frac{yp}{\sqrt{(aa+xx)}},$$

si per $\frac{\partial x}{yp}$ multiplicetur, ob $q\partial x=\partial p$ et $p\partial x=\partial y$, abit in

$$\frac{\alpha \partial p}{p} + \frac{\beta \partial y}{y} = \frac{\partial x}{\sqrt{(aa+xx)}},$$

cujus singuli termini sunt integrabiles. Prodit ergo

$$p^\alpha y^\beta = C[x+\sqrt{(aa+xx)}], \text{ hincque}$$

$$y^{\frac{\beta}{\alpha}} \partial y = C \partial x [x+\sqrt{(aa+xx)}]^{\frac{1}{\alpha}},$$

quae aequatio denuo integrata praebet integrale ante inventum. Forma ergo generalis aequationum hoc modo resolubilium est $P\partial p + Y\partial y + X\partial x = 0$, existente P functione ipsius p, Y ipsius y et X ipsius x, quae nostro more repraesentatur per $Pq + Yp + X = 0$. Hinc ergo perspicitur quomodo etiam aequationes differentio-differentiales ope idonci multiplicatoris ad integrationem perduci queant; quae methodus, cum in aequationibus differentialibus primi gradus insignem usum praestiterit, eo magis excolenda videtur, quod etiam ad aequationes differentiales altiorum graduum pateat, quod argumentum infra fusius pertractare conabimur.

Exemplum 2.

830. *Sumto elemento* ∂x *constante, si proponatur haec aequatio*

$$xy\partial\partial y = y\partial x \partial y + x\partial y^2 + \frac{bx\partial y^2}{\sqrt{(aa-xx)}},$$

ejus integrale invenire.

Posito $\partial y = p\partial x$ et $\partial p = q\partial x$, erit

$$xyq = yp + xpp + \frac{bxpp}{\sqrt{(aa-xx)}},$$

quae facto $p = uy$ et $q = vy$, abit in

$$xv = u + uux + \frac{buux}{\sqrt{(aa-xx)}}:$$

unde oritur haec aequatio differentialis

$$\partial u + uu\partial x = \frac{u\partial x}{x} + uu\partial x + \frac{buu\partial u}{\sqrt{(aa-xx)}}, \text{ seu}$$

$$\frac{x\partial u - u\partial x}{uu} = \frac{bx\partial x}{\sqrt{(aa-xx)}},$$

cujus integrale

$$C - \frac{x}{u} = -b\sqrt{(aa - xx)}, \text{ seu}$$

$$u = \frac{x}{C + b\sqrt{(aa + xx)}}, \text{ ergo } \frac{\partial y}{y} = \frac{x\partial x}{C + b\sqrt{(aa - xx)}}.$$

Statuatur $\sqrt{(aa - xx)} = t$, ut sit $x\partial x = -t\partial t$, erit

$$\frac{\partial y}{y} = \frac{-t\partial t}{C + bt} = -\frac{\partial t}{b} + \frac{C\partial t}{b(C + bt)}, \text{ et}$$

$$ly = -\frac{t}{b} + \frac{C}{bb} l(C + bt) + lc.$$

Sit $C = nbb$, erit

$$l\frac{y}{c} = -\frac{\sqrt{(aa - xx)}}{b} + n\, l\frac{nb + \sqrt{(aa - xx)}}{b}$$

ubi c et n sunt constantes arbitrariae.

CAPUT IV.

DE
AEQUATIONIBUS DIFFERENTIO-DIFFERENTIALIBUS IN QUIBUS ALTERA VARIABILIS UNICAM HABET DIMENSIONEM.

Problema 101.

831.

Sumto elemento ∂x constante, si proponatur aequatio hujus formae $\partial\partial y + P\partial x\partial y + Qy\partial x^2 = 0$, ubi P et Q sint functiones quaecunque ipsius x, eam ad aequationem differentialem primi gradus revocare.

Solutio.

Ponendo $\partial y = p\partial x$ et $\partial p = q\partial x$, aequatio proposita induit hanc formam $q + Pp + Qy = 0$, in qua si methodo ante exposita statuamus $p = uy$ et $q = vy$, obtinebimus hanc inter x, u et v aequationem $v + Pu + Q = 0$, hincque $v = -Pu - Q$. Tum vero fit

$$\partial y = uy\partial x \text{ et } u\partial y + y\partial u = vy\partial x,$$

ita ut sit

$$\frac{\partial y}{y} = u\partial x = \frac{v\partial x - \partial u}{u},$$

ideoque

$$\partial u + uu\partial x + Pu\partial x + Q\partial x = 0,$$

substituto pro v valore.

Qua aequatione resoluta erit $ly = \int u\partial x$. Vel sine his

substitutionibus statim in ipsa aequatione proposita ponamus $y = e^{\int u \partial x}$, unde fit

$$\partial y = e^{\int u \partial x} u \partial x, \text{ et } \partial \partial y = e^{\int u \partial x} (\partial u \partial x + u u \partial x^2),$$

et cum facta substitutione quantitas exponentialis $e^{\int u \partial x}$ ex calculo tollatur, obtinebitur praecedens aequatio differentialis primi gradus

$$\partial u + u u \partial x + P u \partial x + Q \partial x = 0,$$

a cujus resolutione integratio aequationis differentio-differentialis propositae pendet.

Corollarium 1.

832. Haec aequatio differentialis primi gradus pluribus modis in alias formas sibi fere similes transmutari potest. Veluti si ponamus $u = M z$, prodit

$$M \partial z + z (\partial M + P M \partial x) + M M z z \partial x + Q \partial x = 0;$$

ubi pro M ejusmodi functionem ipsius x accipere licet, ut terminus ipsa littera z affectus evanescat, quod fit si

$$\partial M + M P \partial x = 0 \text{ seu } M = C e^{-\int P \partial x}.$$

Corollarium 2.

833. Similis forma prodit ponendo $u = \frac{K}{z}$, fit enim

$$-\frac{K \partial z}{z z} + \frac{\partial K}{z} + \frac{K K \partial x}{z z} + \frac{K P \partial x}{z} + Q \partial x = 0 \text{ seu}$$

$$K \partial z - z (\partial K + K P \partial x) - Q z z \partial x - K K \partial x = 0,$$

ubi secundus terminus, sumendo $K = C e^{-\int P \partial x}$, pariter evanescit.

Corollarium 3.

834. Similis transformatio generalius instituitur, ponendo $u = K + M z$, prodit enim

$$\partial K + M \partial z + z \partial M + K K \partial x + 2 K M z \partial x + M M z z \partial x^2 + K P \partial x + M P z \partial x + Q \partial x = 0.$$

quae ordinata praebet

$$M \partial z + z (\partial M + 2 K M \partial x + M P \partial x) + M M z z \partial x^2 + \partial K + K K \partial x + K P \partial x + Q \partial x = 0;$$

unde secundus terminus tollitur sumendo

$$M = C e^{-\int \partial x (2 K + P)} \text{ seu } K = \frac{-\partial M - M P \partial x}{2 M \partial x}.$$

Corollarium 4.

835. Adhuc generalius similis forma oritur, si ponatur $u = \frac{K + M z}{L + N z}$, unde prodit

$$\partial z (L M - K N) + L \partial K - K \partial L + z (L \partial M - M \partial L + N \partial K - K \partial N) + z z (N \partial M - M \partial N) + (K + M z)^2 \partial x + P (K + M z)(L + N z) \partial x + Q (L + N z)^2 \partial x = 0,$$

quae reducitur ad hanc formam

$$0 = \partial z (L M - K N) + z \left\{ \begin{array}{l} L \partial M - M \partial L + N \partial K - K \partial N \\ + 2 K M \partial x + P (K N + L M) \partial x + 2 L N Q \partial x \end{array} \right\} + z z (N \partial M - M \partial N + M M \partial x + M N P \partial x + N N Q \partial x) + L \partial K - K \partial L + K K \partial x + K L P \partial x + L L Q \partial x:$$

ubi pro K, L, M et N functiones ejusmodi ipsius x accipere licet, ut forma prodeat tractatu facillima.

Scholion.

836. Quoniam hujusmodi aequationes differentio-differentiales, in quibus variabilis y unicam habet dimensionem frequentissime occurrere solent, merito geometrae tantopere in resolvenda aequatione

$$\partial u + u u \partial x + P u \partial x + Q \partial x = 0$$

studium et operam collocarunt, quae etiam forma generaliori ita repraesentari potest

$$\partial z + P z \partial x + R z z \partial x + Q \partial x = 0,$$

cujus quidem casum eximium $\partial z + z z \partial x = a x^n \partial x$ olim Comes *Riccati* in haud spernendum Analyseos incrementum proposuerat. Inter transformationes autem hujus casus praecipue notari meretur positio $x = t^{\frac{2}{n+2}}$, quae dat

$$\partial z + \frac{2}{n+2} z z t^{\frac{-n}{n+2}} \partial t = \frac{2a}{n+2} t^{\frac{n}{n+2}} \partial t;$$

unde ponendo $z = C t^{\frac{n}{n+1}}$, prodit

$$(n+2) C t^{\frac{n}{n+2}} \partial v + C n t^{\frac{-2}{n+2}} v \partial t$$
$$+ 2 C C t^{\frac{n}{n+2}} v v \partial t = 2 a t^{\frac{n}{n+2}} \partial t; \text{ seu}$$
$$(n+2) C \partial v + \frac{n C v \partial t}{t} + 2 C C v v \partial t = 2 a \partial t:$$

ita ut hic nulla potestas indefinita ipsius t occurrat. Si hic porro ponatur $v = \frac{\alpha}{t} + s$, fiet

$$-\frac{(n+2) C \alpha \partial t}{t t} + (n+2) C \partial s + \frac{n C s \partial t}{t} + 2 C C s s \partial t = 2 a \partial t,$$
$$+\frac{n C \alpha \partial t}{t t} \qquad + \frac{4 C C \alpha s \partial t}{t}$$
$$+\frac{2 C C \alpha \alpha \partial t}{t t}$$

ubi si capiatur $\alpha = \frac{-n}{4 C}$, orietur

$$(n+2) C \partial s + 2 C C s s \partial t = 2 a \partial t - \frac{n(n+4) \partial t}{8 t t}$$

quae forma simplicissima videtur.

Theorema.

837. Sumto elemento ∂x constante, si aequationi differentio-differentiali

$$\partial \partial y + P \partial x \partial y + Q y \partial x^2 = 0.$$

satisfaciant integralia particularia $y = M$ et $y = N$, ita ut ratio $M:N$ non sit constans, erit ejus integrale completum $y = \alpha M + \beta N$.

Demonstratio.

Cum valores $y = M$ et $y = N$ satisfaciant aequationi propositae, erit

$$\partial\partial M + P\partial x\partial M + QM\partial x^2 = 0 \text{ et}$$

$$\partial\partial N + P\partial x\partial N + QN\partial x^2 = 0,$$

unde patet, ponendo $y = \alpha M + \beta N$, aequationi quoque satisfieri, cum fiat

$$\left.\begin{array}{l} +\alpha(\partial\partial M + P\partial x\partial M + QM\partial x^2) \\ +\beta(\partial\partial N + P\partial x\partial N + QN\partial x^2) \end{array}\right\} = 0.$$

Quoniam vero hoc integrale $y = \alpha M + \beta N$ duas constantes α et β complectitur, quas pro lubitu definire licet, id completum sit necesse est, nisi forte N sit multiplum ipsius M.

Corollarium 1.

838. Ex datis ergo duobus integralibus particularibus hujusmodi aequationis, ejus integrale completum formari potest, siquidem illa duo integralia sint inter se diversa.

Corollarium 2.

839. Cum posito $y = e^{\int u\partial x}$ seu $u = \frac{\partial y}{y\partial x}$, prodeat

$$\partial u + uu\partial x + Pu\partial x + Q\partial x = 0.$$

Si huic aequationi satisfaciant valores $u = \frac{\partial M}{M\partial x}$ et $u = \frac{\partial N}{N\partial x}$, eidem quoque satisfaciet valor $u = \frac{\alpha\partial M + \beta\partial N}{(\alpha N + \beta M)\partial x}$.

Corollarium 3.

840. Si ergo aequationis $\partial u + uu\partial x + Pu\partial x +$

$Q\partial x = 0$ habeantur duo integralia particularia $u = R$ et $u = S$, ob

$$M = e^{\int R\partial x} \text{ et } N = e^{\int S\partial x},$$

integrale completum erit

$$u = \frac{\alpha e^{\int R\partial x} R + \beta e^{\int S\partial x} S}{\alpha e^{\int R\partial x} + \beta e^{\int S\partial x}} \text{ sive}$$

$$u = R + \frac{\beta e^{\int S\partial x}(S - R)}{\alpha e^{\int R\partial x} + \beta e^{\int S\partial x}}.$$

Scholion.

841. Maximi momenti est haec observatio, quod in hujusmodi aequationibus ex cognitis binis integralibus particularibus integrale completum assignari possit. Plerumque autem cognito uno integrali particulari, in eo signum radicale inesse solet, ob cujus ambiguitatem duo simul integralia particularia innotescunt. Ita si aequationi

$$\partial u + uu\partial x + Pu\partial x + Q\partial x = u$$

satisfaciat valor $u = T + \sqrt{V}$, eidem satisfaciet $u = T - \sqrt{V}$, unde integrale completum erit

$$u = T + \sqrt{V} - \frac{2B\sqrt{V}}{\alpha e^{2\int\partial x\sqrt{V}} + \beta} \text{ seu}$$

$$u = T + \frac{\alpha e^{2\int\partial x\sqrt{V}}\sqrt{V} - \beta\sqrt{V}}{\alpha e^{2\int\partial x\sqrt{V}} + \beta}.$$

Ac si forte sit $\sqrt{V}$ imaginarium, puta $\sqrt{V} = X\sqrt{-1}$. ob

$$e^{\pm\int\partial x\sqrt{V}} = \cos\int X\partial x + \sqrt{-1}.\sin.\int X\partial x, \text{ erit}$$

$$u = T + \frac{(\alpha-\beta)\cos.\int X\partial x + (\alpha+\beta)\sin.\int X\partial x.\sqrt{-1}}{(\alpha+\beta)\cos.\int X\partial x + (\alpha-\beta)\sin.\int X\partial x.\sqrt{-1}} X\sqrt{-1}.$$

seu posito

$$(\alpha - \beta)\sqrt{-1} = \gamma \text{ et } \alpha + \beta = \delta,$$

$$u = T + \frac{\gamma\cos.\int X\partial x - \delta\sin.\int X\partial x}{\delta\cos.\int X\partial x + \gamma\sin.\int X\partial x}.X, \text{ vel etiam}$$

$$u = T + X\,\text{tang.}(\int X\partial x + \zeta).$$

Problema 102.

842. Sumto elemento ∂x constante, invenire integrale completum hujus aequationis differentio-differentialis

$$\partial \partial y + A \partial y \partial x + B y \partial x^2 = 0.$$

Solutio.

Posito $y = e^{\int u \partial x}$ prodit haec aequatio

$$\partial u + u u \partial x + A u \partial x + B \partial x = 0, \text{ sive}$$

$$\partial x = \frac{-\partial u}{u u + A u + B},$$

cui satisfit tribuendo ipsi u ejusmodi valorem constantem ut evadat $u u + A u + B = 0$, qui sunt

$$u = -\tfrac{1}{2} A \pm \sqrt{(\tfrac{1}{4} A A - B)}.$$

Hinc ergo cum habeantur bina integralia particularia $y = e^{\int u \partial x}$, posito $\sqrt{(\tfrac{1}{4} A A - B)} = n$, erit integrale completum

$$y = e^{-\frac{1}{2} A x} (\alpha e^{n x} + \beta e^{-n x}),$$

ac si sit n numerus imaginarius, puta $n = m\sqrt{-1}$, erit

$$y = e^{-\frac{1}{2} A x} (\alpha \cos. m x + \beta \sin. m x) = C e^{-\frac{1}{2} A x} \sin. (m x + \gamma).$$

Sin autem sit $n = 0$, prodibit

$$y = e^{-\frac{1}{2} A x} (\alpha + \beta x).$$

Corollarium 1.

843. Ad integrale ergo aequationis propositae inveniendum, resolvi oportet aequationem algebraicam $u u + A u + B = 0$, quae oritur ex proposita

$$\partial \partial y + A \partial y \partial x + B y \partial x^2 = 0,$$

si loco y, ∂y, $\partial \partial y$, scribatur u^0, u^1, u^2, et elementum ∂x reji-

ciatur, tum enim binae radices illius aequationis dabunt integrale completum.

Corollarium 2.

844. Scilicet si aequationis $uu + Au + B = 0$ factores sint $(u + f)(u + g)$, ob valores $u = -f$ et $u = -g$, integrale completum erit $y = \alpha e^{-fx} + \beta e^{-gx}$. At si sit $g = f$, erit $y = e^{-fx}(\alpha + \beta x)$.

Corollarium 3.

845. Si aequatio $uu + Au + B = 0$ habeat factores imaginarios, quo casu hujusmodi formam habebit

$$uu + 2fu\cos.\zeta + ff = 0, \text{ erit } u = -f\cos.\zeta \pm f\sqrt{-1}.\sin.\zeta,$$

hincque integrale completum erit:

$$y = e^{-fx\cos.\zeta}(\alpha\cos.fx\sin.\zeta + \beta\sin.fx\sin.\zeta), \text{ sive}$$
$$y = C e^{-fx\cos.\zeta}\sin.(fx\sin.\zeta + \gamma).$$

Scholion.

846. Idem integrale completum reperitur methodo consueta ex aequatione $\partial x = \frac{-\partial u}{uu + Au + B}$: posito enim

$$uu + Au + B = (u + f)(u + g), \text{ erit}$$
$$(g - f)\partial x = \frac{\partial u}{u + g} - \frac{\partial u}{u + f}, \text{ et } C e^{(g-f)x} = \frac{u + g}{u + f};$$

unde fit

$$u = \frac{g - Cfe^{(g-f)x}}{Ce^{(g-f)x} - 1}, \text{ seu}$$
$$u = \frac{-\alpha f e^{gx} + \beta g e^{fx}}{\alpha e^{gx} - \beta e^{fx}}.$$

Tum vero fit

$$\int u\partial x = -\int \frac{\alpha f e^{(g-f)x} - \beta g}{\alpha e^{(g-f)x} - \beta}\partial x = -\int \frac{u\partial u}{(u + f)(u + g)};$$

ideoque

$$\int u\,\partial x = \frac{f}{g-f}\,l(u+f) - \frac{g}{g-f}\,l(u+g).$$

Hinc

$$y = e^{\int u\,\partial x} = C\,(u+f)^{\frac{f}{g-f}}\,(u+g)^{\frac{-g}{g-f}}.$$

At est

$$u+f = \frac{\beta\,(g-f)\,e^{fx}}{\alpha\,e^{gx} - \beta\,e^{fx}} \text{ et } u+g = \frac{\alpha\,(g-f)\,e^{gx}}{\alpha\,e^{gx} - \beta\,e^{fx}},$$

unde colligitur mutando constantem C

$$y = \frac{C\,e^{\frac{ffx}{g-f}}\,e^{\frac{-ggx}{g-f}}}{(\alpha\,e^{gx} - \beta\,e^{fx})^{\frac{f-g}{g-f}}} = C\,e^{-(f+g)x}\,(\alpha\,e^{gx} - \beta\,e^{fx}),$$

seu $y = \alpha\,e^{-fx} + \beta\,e^{-gx}$, ut ante. Hinc ergo patet, quantum subsidium afferat formatio integralis completi ex binis particularibus.

Problema 103.

847. Sumto elemento ∂x constante, si proponatur haec aequatio differentio-differentialis

$$\partial\partial y + \frac{A\,\partial y\,\partial x}{x} + \frac{B\,y\,\partial x^2}{xx} = 0,$$

ejus integrale completum invenire.

Solutio.

Ponatur $\partial y = p\,\partial x$ et $\partial p = q\,\partial x$, ut habeamus

$$q + \frac{Ap}{x} + \frac{By}{xx} = 0, \text{ seu } q = \frac{-Ap}{x} - \frac{By}{xx}.$$

Sit nunc $p = \frac{uy}{x}$, erit $\partial y = \frac{uy\,\partial x}{x}$, et

$$\partial p = \frac{u\,\partial y + y\,\partial u}{x} - \frac{uy\,\partial x}{xx} = -\frac{A\,\partial y}{x} - \frac{By\,\partial x}{xx},$$

unde colligimus

$$\frac{\partial y}{y} = \frac{u\,\partial x}{x} = \frac{u\,\partial x - B\,\partial x - x\,\partial u}{xu + Ax}, \text{ seu}$$

$$x\partial u + B\partial x + uu\partial x + (A-1)u\partial x = 0,$$

hincque $\frac{\partial x}{x} = \frac{-\partial u}{uu+(A-1)u+B}$; cui particulariter satisfit ponendo $uu+(A-1)u+B=0$.

Sit primo $uu+(A-1)u+B=(u+f)(u+g)$, erit particulariter $ly=-flx$ et $y=x^{-f}$, similique modo $y=x^{-g}$, unde integrale completum erit

$$y = \alpha x^{-f} + \beta x^{-g}.$$

Si sit $g=f$, statuatur $g=f-\omega$ evanescente ω, erit

$$x^{-g} = x^{-f}.x^{\omega} = x^{-f}(1+\omega lx),$$

ergo hoc casu fit

$$y = x^{-f}(\alpha + \beta lx).$$

Sit denique

$$uu+(A-1)u+B = uu+2fu\cos.\zeta+ff, \text{ erit}$$

$$u = -f(\cos.\zeta \pm \sqrt{-1}.\sin.\zeta),$$

ergo particulariter

$$y = x^{-f\cos.\zeta}.x^{\pm f\sqrt{-1}.\sin.\zeta} = x^{-f\cos.\zeta}[\cos.(f\sin.\zeta.lx) \pm \sqrt{-1}.\sin.(f\sin.\zeta.lx)],$$

quare integrale completum erit

$$y = Cx^{-f\cos.\zeta}\sin.(f\sin.\zeta.lx+\gamma).$$

Corollarium 1.

848. Hujus ergo aequationis

$$\partial\partial y + (f+g+1)\frac{\partial y\partial x}{x} + \frac{fgy\partial x^2}{xx} = 0,$$

integrale completum est

$$y = \alpha x^{-f} + \beta x^{-g}.$$

Hujus autem

$$\partial\partial y + (2f+1)\frac{\partial y\partial x}{x} + \frac{ffy\partial x^2}{xx} = 0,$$

integrale completum est

$$y = x^{-f}(\alpha + \beta\, l x).$$

Corollarium 2.

819. At si aequatio proposita hujusmodi formam habuerit

$$\partial\partial y + (1 + 2f\cos.\zeta)\frac{\partial y\,\partial x}{x} + \frac{ffy\,\partial x}{xx} = 0,$$

tum ejus integrale completum erit

$$y = C\, x^{-f\cos.\zeta}\sin.(f\sin.\zeta\, l x + \gamma).$$

Scholion.

820. Similem resolutionem quoque admittit haec aequatio differentio-differentialis

$$\partial\partial y - \frac{n\,\partial y\,\partial x}{x} + A\,x^n\,\partial y\,\partial x + B\,x^{2n}\,y\,\partial x^2 = 0.$$

Ponatur enim $\partial y = x^n y\,u\,\partial x$, et cum sit

$$\partial\partial y = x^n y\,\partial x\,\partial u + n\,x^{n-1}\,y u\,\partial x^2 + x^{2n}\,y u^2\,\partial x^2,$$

erit per y dividendo

$$x^n\partial x\partial u + n x^{n-1} u\,\partial x^2 + x^{2n} u u\,\partial x^2 - n x^{n-1} u\,\partial x^2 + A x^{2n} u\,\partial x^2 + B x^{2n}\,\partial x^2 = 0,$$

hinc

$$\partial u + x^n\,u u\,\partial x + A\,x^n\,u\,\partial x + B\,x^n\,\partial x = 0,$$

ideoque

$$x^n\,\partial x = \frac{-\partial u}{uu + Au + B},$$

cui particulariter satisfit ponendo $uu + A u + B = 0$, unde u duplicem consequitur valorem constantem quorum alter sit $u = -f$ alter $u = -g$. Quocirca integralia particularia erunt

$$y = e^{\frac{-f x^{n+1}}{n+1}} \quad \text{et} \quad y = e^{\frac{-g x^{n+1}}{n+1}}.$$

**

Sit brevitatis gratia $\frac{x^{n+1}}{n+1} = t$, erit integrale completum

$$y = \alpha e^{-ft} + \beta e^{-gt},$$

pro casu scilicet

$$uu + Au + B = (u + f)(u + g).$$

At pro casu

$$uu + Au + B = (u + f)^2, \text{ erit } y = e^{-ft}(\alpha + \beta t).$$

Casu autem quo

$$uu + Au + B = uu + 2fu \cos. \zeta + ff, \text{ erit}$$

$$y = C e^{-ft \cos. \zeta} \sin. (ft \sin. \zeta + \gamma).$$

Haec integratio adeo ad hanc formam extendi potest, sumendo pro X functionem quamcunque ipsius x

$$\partial\partial y - \frac{\partial x \partial y}{X} + AX \partial y \partial x + BXXy \partial x^2 = 0.$$

Posito enim $\partial y = Xuy\partial x$ seu $\frac{\partial y}{y} = Xu\partial x$, fit

$$X\partial x = \frac{-\partial u}{uu + Au + B},$$

unde posito $\int X \partial x = t$, integrale completum se habebit ut ante. Scilicet

1) si $A = f + g$ et $B = fg$, erit integrale
$$y = \alpha e^{-ft} + \beta e^{-gt},$$
2) si $A = 2f$ et $B = ff$, erit integrale
$$y = e^{-ft}(\alpha + \beta t),$$
3) si $A = 2f \cos. \zeta$ et $B = ff$, erit integrale
$$y = C e^{-ft \cos. \zeta} \sin. (ft \sin. \zeta + \gamma).$$

Problema 104.

851. Sumto elemento ∂x constante, si P, Q et X denotent functiones quascunque ipsius x, integrationem hujus aequationis differentio-differentialis

$$\partial\partial y + P\,\partial y\,\partial x + Q\,y\,\partial x^2 = X\,\partial x^2,$$

ad aequationem differentialem primi ordinis reducere.

Solutio.

Hic singulari modo procedamus, loco y binas novas incognitas introducendo. Statuatur scilicet $y = uv$, et cum sit

$$\partial y = u\,\partial v + v\,\partial u \text{ et } \partial\partial y = u\,\partial\partial v - 2\,\partial u\,\partial v + v\,\partial\partial u,$$

aequatio nostra induet hanc formam

$$u\,\partial\partial v + 2\,\partial u\,\partial v + v\,\partial\partial u + P\,u\,\partial x\,\partial v + P\,v\,\partial x\,\partial u + Q\,u\,v\,\partial x^2 = X\,\partial x^2.$$

Jam altera v ita determinetur, ut termini ipsa littera u affecti destruantur, quod fit si

$$\partial\partial v + P\,\partial x\,\partial v + Q\,v\,\partial x^2 = 0,$$

unde per superiora v per x determinetur, quo facto superest haec aequatio

$$2\,\partial u\,\partial v + v\,\partial\partial u + P\,v\,\partial x\,\partial u = X\,\partial x^2,$$

unde cum v jam detur per x, quantitas u definiri debet. Ponatur $\partial u = s\,\partial x$, eritque

$$v\,\partial s + 2\,s\,\partial v + P\,s\,v\,\partial x = X\,\partial x,$$

quae multiplicata per $v\,e^{\int P\,\partial x}$ integrabilis redditur; prodit enim

$$v\,v\,s\,e^{\int P\,\partial x} = \int e^{\int P\,\partial x} X\,v\,\partial x, \text{ ideoque}$$

$$y = \frac{e^{-\int P\,\partial x}}{v\,v} \int e^{\int P\,\partial x} X\,v\,\partial x \text{ et}$$

$$u = \int \frac{e^{-\int P\,\partial x}\,\partial x}{v\,v} \int e^{\int P\,\partial x} X\,v\,\partial x.$$

Quare cum incognita v fuerit determinata aequatione

$$\partial\partial v - P\,\partial x\,\partial v + Q\,v\,\partial x^2 = 0,$$

integrale aequationis propositae erit

$$y = v\int \frac{e^{-\int P\partial x}\partial x}{vv}\int e^{\int P\partial x} X v\partial x.$$

Corollarium 1.

852. Ut integratio ad aequationem differentialem primi ordinis revocetur, ponatur $v = e^{\int t\partial x}$, et quantitas t definietur per hanc aequationem

$$\partial t + tt\partial x + Pt\partial x + Q\partial x = 0,$$

quo facto integrale quaesitum erit

$$y = e^{\int t\partial x}\int e^{-\int (P+2t)\partial x}\partial x\int e^{\int (P+t)\partial x} X\partial x.$$

Corollarium 2.

853. Cum sit $(P+t)\partial x = -\frac{\partial t}{t} - \frac{Q\partial x}{t}$, erit

$$e^{\int (P+t)\partial x} = \frac{1}{t} e^{-\int\frac{Q\partial y}{t}}, \text{ hincque}$$

$$y = e^{\int t\partial x}\int e^{\int\frac{Q\partial x}{t} - \int t\partial x} t\partial x\int e^{-\int\frac{Q\partial x}{t}} X\frac{\partial x}{t},$$

ubi duplex integratio ad integrale completum perducit.

Scholion 1.

854. Alio modo qui propius ad ante usitatum accedat, eadem integratio institui potest. Ponatur scilicet pro aequatione proposita $\partial y = ty\partial x + v\partial x$, ubi v certam ipsius x functionem designet ex functione X determinandam. Cum igitur sit

$$\partial\partial y = y\partial t\partial x + (ty\partial x + v\partial x) + \partial v\partial x,$$

erit facta substitutione

$$\left.\begin{matrix} y\partial t\partial x + tty\partial x^2 + Pty\partial x^2 + Qy\partial x^2 \\ + tv\partial x^2 + \partial v\partial x + Pv\partial x^2 + X\partial x^2 \end{matrix}\right\} = 0,$$

cujus aequationis utraque pars, tum ea quae per y multiplicatur,

quam altera ab y libera, seorsim nihilo aequetur, unde has duas aequationes nanciscimur;

$$\partial t + tt\partial x + Pt\partial x + Q\partial x = 0 \text{ et}$$

$$\partial v + tv\partial x + Pv\partial x = X\partial x,$$

ex quarum illa t per x ut ante definiri debet, tum vero erit ex ista

$$e^{\int (P+t)\partial x} v = \int e^{\int (P+t)\partial x} X\partial x.$$

Jam vero ex aequatione assumta $\partial y - ty\partial x = v\partial x$ colligitur

$$e^{-\int t\partial x} y = \int e^{-\int t\partial x} v\partial x,$$

ubi si loco v valor modo inventus substituatur, praecedens integralis forma obtinetur.

Scholion 2.

855. Ex hac operatione sequi videtur, aequationis propositae

$$\partial\partial y + P\partial y\partial x + Qy\partial x^2 = X\partial x^2,$$

integrationem necessario pendere ab integratione hujus

$$\partial\partial v + P\partial v\partial x + Qv\partial x^2 = 0,$$

quandoquidem hac concessa illa exhiberi potest. Minime tamen hinc vicissim colligere licet, si posterioris resolutio vires nostras superet, etiam priorem nullo modo integrari posse, quin potius facile est infinitos casus exhibere, quibus prior integrationem admittat, cum tamen posterior irresolubilis existat. Sit enim $P = 0$ ut $Q = ax$, atque certum est aequationem posteriorem $\partial\partial v + axv\partial x^2 = 0$, nulla adhuc methodo resolvi posse, cum posito $v = e^{\int t\partial x}$, abeat in

$$\partial t + tt\partial x + ax\partial x = 0;$$

neque tamen hinc sequitur aequationem priorem

$$\partial\partial y + axy\partial x^2 = X\partial x^2,$$

semper esse intractabilem. Infiniti enim casus pro X assignari possunt, quibus integratio succedat. Sumta enim pro y functione quacunque ipsius x, reperietur pro X ejusmodi functio, ut aequationi

valor pro y assumtus satisfaciat. Veluti posito $y=\frac{\beta x}{\alpha}$, ob $\partial\partial y=0$ fit $X=\beta xx$, atque aequationi

$$\partial\partial y+\alpha xy\partial x^2=\beta xx\partial x^2$$

utique satisfacit integrale $y=\frac{\beta x}{\alpha}$. Interim tamen hoc integrale tantum est particulare, ac dubium adhuc relinquitur, an etiam integrale completum exhiberi possit. At posito $y=\frac{\beta x}{\alpha}+z$, pro integrali completo inveniendo prodit $\partial\partial z+\alpha xz\partial x^2=0$, quae cum resolutionem respuat, evidens est integrale completum etiam in genere exhiberi non posse, nisi simul altera aequatio integrationem admittat.

Problema 105.

656. Sumto elemento ∂x constante, invenire integrale completum hujus aequationis differentio-differentialis

$$\partial\partial y+A\partial y\partial x+By\partial x^2=X\partial x^3$$

denotante X functionem quamcunque ipsius x.

Solutio.

Posito $y=uv$, aequatio proposita in duas sequentes resolvitur

$$\partial\partial v+A\partial v\partial x+Bv\partial x^2=0 \text{ et}$$

$$v\partial\partial u+2\partial v\partial u+Av\partial x\partial u=X\partial x^2.$$

Quodsi ergo ex priori valor ipsius v per x definiatur, integrale completum ex posteriore ita se habebit, ut ob $P=A$ sit

$$y=v\int\frac{e^{-Ax}\partial x}{vv}\int e^{Ax}Xv\partial x,$$

ubi cum duplex integratio integrale completum producat, sufficiet

pro v integrale particulare prioris aequationis assumsisse, id quod etiam ex solutione generali patebit. Cum igitur pro resolutione prioris aequationis formanda sit haec aequatio quadratica $tt + At + B = 0$, pro ejus indole tres casus evolvi conveniet.

I. Si $tt + At + B = (t + f)(t + g)$, ut sit $A = f + g$ et $B = fg$, erit $v = \alpha e^{-fx} + \beta e^{-gx}$. Sumatur primo tantum integrale particulare $v = e^{-fx}$, et ob $A = f + g$, fiet

$$y = e^{-fx} \int e^{(f-g)x} \partial x \int e^{gx} X \partial x:$$

sit $e^{(f-g)x} \partial x = \partial R$, et $\int e^{gx} X \partial x = S$, ut fiat

$$y = e^{-fx} \int S \partial R = e^{-fx} (RS - \int R \partial S);$$

at est $R = \frac{1}{f-g} e^{(f-g)x}$, unde colligitur

$$y = \frac{1}{f-g} e^{-gx} S - \frac{1}{f-g} e^{-fx} \int e^{fx} X \partial x, \text{ sive}$$

$$(f-g) y = e^{-gx} \int e^{gx} X \partial x - e^{-fx} \int e^{fx} X \partial x,$$

quod idem integrale prodiisset, si altero particulari $v = e^{-gx}$ usi essemus.

In genere autem sumto $v = \alpha e^{-fx} + \beta e^{-gx}$, statuatur ut ante

$$\frac{e^{-(f+g)x} \partial x}{vv} = \partial R, \text{ et } \int e^{(f+g)x} X v \partial x = S,$$

ut sit pariter

$$y = v \int S \partial R = v (RS - \int R \partial S).$$

Fingamus

$$R = \frac{C e^{\lambda x}}{v}, \text{ et ob } \partial v = - \partial x (\alpha f e^{-fx} + \beta g e^{-gx}),$$

erit

$$\partial R = \frac{C e^{\lambda x} \partial x (\alpha \lambda e^{-fx} + \beta \lambda e^{-gx} + \alpha f e^{-fx} + \beta g e^{-gx})}{vv}:$$

Sit jam $\lambda = -g$, et $C\alpha(f-g) = 1$, ut ∂R datum adipiscatur valorem, ob $C = \frac{1}{\alpha(f-g)}$, erit

$$R = \frac{e^{-gx}}{\alpha(f-g)v}, \text{ et } R\partial S = \frac{1}{\alpha(f-g)} e^{fx} X \partial x;$$

tum vero

$$S = \alpha \int e^{gx} X \partial x + \beta \int e^{fx} X \partial x,$$

unde conficitur

$$y = v\left(\frac{e^{-gx}}{(f-g)v}\int e^{gx} X\partial x + \frac{\beta e^{-gx}}{\alpha(f-g)v}\int e^{fx} X\partial x - \frac{1}{\alpha(f-g)}\int e^{fx} X\partial x\right),$$

seu prorsus ut ante

$$(f-g)y = e^{-gx}\int e^{gx} X\partial x - e^{-fx}\int e^{fx} X \partial x.$$

II. Si $tt + At + B = (t+f)^2$, seu $A = 2f$ et $B = ff$, erit ex priori aequatione $v = e^{-fx}(\alpha + \beta x)$. Ponatur ut ante

$$\frac{e^{-2fx}\partial x}{vv} = \partial R, \text{ et } \int e^{2fx} X v \partial x = S,$$

ut habeatur $y = v(RS - \int R\partial S)$. Cum ergo sit

$$\partial R = \frac{\partial x}{(\alpha+\beta x)^2}, \text{ erit } R = -\frac{1}{\beta(\alpha+\beta x)} = -\frac{e^{-fx}}{\beta v}, \text{ et}$$

$$S = \alpha\int e^{fx} X\partial x + \beta\int e^{fx} X x\partial x = \int e^{fx} X\partial x(\alpha+\beta x),$$

quare

$$vRS = -\frac{\alpha}{\beta} e^{-fx}\int e^{fx} X\partial x - e^{-fx}\int e^{fx} X x\partial x, \text{ et}$$

$$\int R\partial S = -\frac{1}{\beta}\int e^{fx} X\partial x,$$

unde conficitur

$$y = e^{-fx} x\int e^{fx} X\partial x - e^{-fx}\int e^{fx} X x\partial x;$$

seu cum sit

$$\partial . e^{fx} y = \partial x \int e^{fx} X \partial x,$$

erit succinctius

$$y = e^{-fx} \int \partial x \int e^{fx} X \partial x.$$

III. Si $tt + At + B = tt + 2ft \cos.\zeta + ff$, seu $A = 2f \cos.\zeta$ et $B = ff$, erit

$$v = e^{-fx \cos.\zeta} \sin.(fx \sin.\zeta + \gamma).$$

Ponatur

$$\frac{e^{-2fx\cos.\zeta}\partial x}{vv} = \frac{\partial x}{\sin.(fx \sin.\zeta + \gamma)^2} = \partial R \text{ et}$$

$$e^{2fx\cos.\zeta} X v \partial x = e^{fx\cos.\zeta} X \partial x \sin.(fx \sin.\zeta + \gamma) = \partial S,$$

ut obtineatur $y = vRS - v\int R\partial S$. At est

$$R = -\frac{1}{f \sin.\zeta} \cdot \frac{\cos.(fx \sin.\zeta + \gamma)}{\sin.(fx \sin.\zeta + \gamma)};$$

hincque

$$vRS = -\frac{1}{f \sin.\zeta} e^{-fx\cos.\zeta} \cos.(fx\sin.\zeta + \gamma) \int e^{fx\cos.\zeta} X \partial x \sin.(fx \sin.\zeta + \gamma)$$

et

$$\int R \partial S = -\frac{1}{f \sin.\zeta} \int e^{fx\cos.\zeta} X \partial x \cos.(fx \sin.\zeta + \gamma).$$

Quocirca obtinebitur

$$fy \sin.\zeta = + e^{-fx\cos.\zeta} \sin.(fx \sin.\zeta + \gamma) \int e^{fx\cos.\zeta} X \partial x \cos.(fx \sin.\zeta + \gamma)$$
$$- e^{-fx\cos.\zeta} \cos.(fx \sin.\zeta + \gamma) \int e^{fx\cos.\zeta} X \partial x \sin.(fx \sin.\zeta + \gamma).$$

Corollarium 1.

857. In hoc postremo integrali si ponamus $fx \sin.\zeta = \Phi$, erit

$$f e^{fx\cos.\zeta} y \sin.\zeta =$$
$$(\sin.\gamma \cos.\Phi + \cos.\gamma \sin\Phi) \int e^{fx\cos.\zeta} X \partial x (\cos.\gamma \cos.\Phi - \sin.\gamma \sin.\Phi)$$
$$+ (\sin.\gamma \sin.\Phi - \cos.\gamma \sin.\Phi) \int e^{fx\cos.\zeta} X \partial x (\sin.\gamma \cos.\Phi + \cos.\gamma \sin.\Phi),$$

seu

$$\int e^{f x \cos.\zeta} y \sin.\zeta =$$
$$+\sin.\gamma\cos.\gamma\cos.\Phi\int e^{f x\cos.\zeta}X\partial x\cos.\Phi - \sin.\gamma^2\cos.\Phi\int e^{f x\cos.\zeta}X\partial x\sin.\Phi$$
$$+\cos.\gamma^2\sin.\Phi\int e^{f x\cos.\zeta}X\partial x\cos\Phi - \sin.\gamma\cos.\gamma\sin.\Phi\int e^{f x\cos.\zeta}X\partial x\sin.\Phi$$
$$+\sin.\gamma^2\sin.\Phi\int e^{f x\cos.\zeta}X\partial x\cos.\Phi + \sin.\gamma\cos.\gamma\sin.\Phi\int e^{f x\cos.\zeta}X\partial x\sin.\Phi$$
$$-\sin.\gamma\cos.\gamma\cos.\Phi\int e^{f x\cos.\zeta}X\partial x\cos.\Phi - \cos.\gamma^2\cos.\Phi\int e^{f x\cos.\zeta}X\partial x\sin.\Phi$$

unde patet angulum γ prorsus ex calculo excedere, fit enim

$$\int e^{f x\cos.\zeta} y\sin.\zeta = \sin.\Phi\int e^{f x\cos.\zeta}X\partial x\cos.\Phi - \cos.\Phi\int e^{f x\cos.\zeta}X\partial x\sin.\Phi.$$

Corollarium 2.

858. Cum igitur loco unius aequationis duas formaverimus integrandas, vidimus sufficere, si alterius integrale saltem particulare fuerit cognitum. In ambobus enim praecedentibus casibus constantes α et β integrale completum praebentes ex calculo sponte evanuerunt, et casu tertio constans γ itidem excessit.

Exemplum.

859. *Sumto elemento ∂x constante, invenire integrale hujus aequationis*

$$\partial\partial y + A\partial y\partial x + By\partial x^2$$
$$= \partial x^2\,[n(n-1)x^{n-2} + nAx^{n-1} + Bx^n].$$

Hoc exemplum ita est comparatum, ut ei manifesto satisfaciat valor $y = x^n$, qui ejus integrale particulare constituit. Ad completum ergo inveniendum, sit $A = f + g$ et $B = fg$; et cum sit

$$X = fg\,x^n + n(f+g)\,x^{n-1} + n(n-1)\,x^{n-2},\ \text{erit}$$
$$\int e^{gx}X\partial x = f e^{gx}x^n + n e^{gx}x^{n-1} + \alpha,\ \text{et}$$
$$\int e^{fx}X\partial x = g\,e^{fx}x^n + n e^{fx}x^{n-1} + \beta,$$

unde ex forma inventa prodit integrale completum

$$(f-g)y = fx^n + nx^{n-1} + \alpha e^{-gx} - gx^n - nx^{n-1} - \beta e^{-fx},$$

seu $y = x^n + \frac{\alpha}{f-g} e^{-gx} - \frac{\beta}{f-g} e^{-fx}$,

vel mutata constantium forma

$$y = x^n + \alpha e^{-fx} + \beta e^{-gx}.$$

Si sit $g = f$, ponatur $g = f + \omega$, existente $\omega = 0$, et ob $e^{-gx} = e^{-fx} . e^{-\omega x} = e^{-fx}(1 - \omega x)$, unde pro $\alpha + \beta$ et $-\beta\omega$ scribendo α et β, erit

$$y = x^n + e^{-fx}(\alpha + \beta x).$$

Sin autem sit

$$f = a + b\sqrt{-1} \text{ et } g = a - b\sqrt{-1},$$

cum fiat

$$y = x^n + e^{-ax}(\alpha e^{-bx\sqrt{-1}} + \beta e^{bx\sqrt{-1}}) \text{ ob}$$
$$e^{\pm bx\sqrt{-1}} = \cos. bx \pm \sqrt{-1} . \sin. bx,$$

mutata forma constantium habebimus

$$y = x^n + e^{-ax}(\alpha \cos. bx + \beta \sin. bx).$$

Scholion.

860. In genere autem si hujusmodi aequationis

$$\partial\partial y + A\partial y\partial x + By\partial x^2 = X\partial x^2,$$

constet integrale particulare, seu valor ipsi satisfaciens $y = t$, integrale completum facile reperitur ponendo $y = t + z$. Cum enim per hypothesin sit

$$\partial\partial t + A\partial t\partial x + Bt\partial x^2 = X\partial x^2,$$

facta hac substitutione orietur

$$\partial\partial z + A\partial z\partial x + Bz\partial x^2 = 0,$$

unde si $A = f + g$ et $B = fg$, colligitur

$$z = \alpha e^{-fx} + \beta e^{-gx},$$

sicque integrale completum erit

$$y = t + \alpha e^{-fx} + \beta e^{-gx},$$

quemadmodum etiam in exemplo allato invenimus.

Problema 106.

861. Sumto elemento ∂x constante, si proponatur haec aequatio differentio-differentialis

$$\partial\partial y - \frac{n\,\partial y\,\partial x}{x} + A x^n \partial y \partial x + B x^{2n} y \partial x^2 = X \partial x^2,$$

existente X functione quacunque ipsius x, ejus integrale completum investigare.

Solutio.

Resolutio hujus aequationis ut supra ex positione $y = uv$ derivari posset, sed alia methodo hic utentes levi substitutione eam ad formam problematis praecedentis reducamus. Scilicet ponamus $x^n \partial x = \partial t$, ut sit $x^{n+1} = (n+1)\,t$, qua substitutione functio X abeat in T, functionem quandam ipsius t. Ne autem assumtio elementi ∂x constantis turbet, hanc conditionem tollamus ponendo $\partial y = p\,\partial x$ et $\partial p = q\,\partial x$, habebimusque

$$q - \frac{n\,p}{x} + A\,x^n p + B\,x^{2n}\,y = X.$$

Cum nunc sit $\partial x = \frac{\partial t}{x^n}$, erit $p = \frac{x^n\,\partial y}{\partial t}$, hincque sumto elemento ∂t constante

$$\partial p = \frac{n\,x^{n-1}\,\partial x\,\partial y}{\partial t} + \frac{x^n\,\partial\partial y}{\partial t} = q\,\partial x = \frac{q\,\partial t}{x^n},$$

ergo

$$q = \frac{n\,x^{n-1}\,\partial y}{\partial t} + \frac{x^{2n}\partial\partial y}{\partial t^2},$$

sicque nostra aequatio erit

$$\frac{x^{2n}\partial\partial y}{\partial t^2} = \frac{n x^{n-1}\partial y}{\partial t} - \frac{n x^{n-1}\partial y}{\partial t} + \frac{A x^{2n}\partial y}{\partial t} + B x^{2n} y = X.$$

Sit $X x^{-2n} = \Theta$, quae quantitas, posito $x^{n+1} = (n+1)\,t$, ut functio ipsius t spectari potest, sicque fiet

$$\partial\partial y + A\,\partial y\,\partial t + B\,y\,\partial t^2 = \Theta\,\partial t^2,$$

in qua aequatione elementum ∂t constans est assumtum, cujus ergo integrale per superiora datur.

I. Si $A = f + g$ et $B = fg$, erit integrale

$$(f - g)\, y = e^{-gt} \int e^{gt} \Theta \partial t - e^{-ft} \int e^{ft} \Theta \partial t,$$

ubi restitutis valoribus $\partial t = x^n \partial x$ et $\Theta = X x^{-2n}$, retento brevitatis gratia $t = \frac{1}{n+1} x^{n+1}$, valor ipsius y ita per x exprimetur

$$(f - g) y = e^{-gt} \int e^{gt}\, x^{-n} X \partial x - e^{-ft} \int e^{ft}\, x^{-n} X \partial x.$$

II. Si $A = 2f$ et $B = ff$, erit integrale

$$y = e^{-ft}\, t \int e^{ft} \Theta \partial t - e^{-ft} \int e^{ft} \Theta t \partial t, \text{ seu}$$

$$y = e^{-ft} \int \partial t \int e^{ft} \Theta \partial t,$$

quod ergo per x ita exprimetur

$$y = e^{-ft} \int x^n \partial x \int e^{ft} x^{-n} X \partial x.$$

III. Si denique $A = 2f \cos. \zeta$ et $B = ff$, erit integrale $f e^{ft \cos \zeta} y \sin. \zeta = \sin. \Phi \int e^{ft \cos. \zeta} \Theta \partial t \cos. \Phi - \cos. \Phi \int e^{ft \cos. \zeta} \Theta \partial t \sin. \Phi$, existente $\Phi = f t \sin. \zeta$; seu $\Phi = \frac{f \sin. \zeta}{n+1} x^{n+1}$, ob $t = \frac{1}{n+1} x^{n+1}$. Quare aequationis propositae integrale erit

$$f e^{ft \cos. \zeta} y \sin. \zeta = \sin. \Phi \int e^{ft \cos. \zeta} x^{-n} X \partial x \cos. \Phi - \cos. \Phi \int e^{ft \cos. \zeta} x^{-n} X \partial x \sin. \Phi$$

Corollarium 1.

862. Si $n = 0$, aequatio proposita abit in eam ipsam, quam problemate praecedente tractavimus, fitque $t = x$, unde etiam integrale eodem redit.

Corollarium 2.

863. Sin autem sit $n = -1$, aequatio nostra fit

$$\partial\partial y + (A + 1) \frac{\partial y \partial x}{x} + \frac{B y \partial x^2}{xx} = X \partial x^2,$$

ubi ergo erit $t = lx$ et $e^{\lambda t} = x^{\lambda}$, tum vero pro casu tertio angulus $\Phi = f \sin. \zeta . lx$.

Scholion.

864. Methodus qua hic usi sumus, hujusmodi aequationes differentiales integrandi, haud satis naturalis videtur, cum ad has quasi solas formas sit adstricta. Quoniam igitur in aequationibus differentialibus primi gradus inventio factorum, quibus eae per se integrabiles reddantur, insignem fructum polliceri videbatur, ejus quoque usum in aequationibus differentialibus secundi gradus ostendere conemur. Hic quidem nihil tam absolutum expectare licet, quod ad omnes omnino aequationum formas pateat, sed quantillum etiam praestare potuerimus, id haud contemnendum Analyseos incrementum spectari debebit. Hac autem methodo eas potissimum aequationes differentiales, in quibus altera variabilis y cum suis differentialibus unam dimensionem nusquam transgreditur, satis commode tractare licet, hincque via perspicietur, quomodo eam magis excoli oporteat.

CAPUT V.

DE

INTEGRATIONE AEQUATIONUM DIFFERENTIALIUM SECUNDI GRADUS, IN QUIBUS ALTERA VARIABILIS UNAM DIMENSIONEM NON SUPERAT, PER FACTORES.

Problema 107.

865.

Sumto elemento ∂x constante, si proponatur haec aequatio:

$$\partial \partial y + A \partial x \partial y + B y \partial x^2 = X \partial x^2,$$

ubi X denotat functionem quamcunque ipsius x, invenire functionem ipsius x, per quam haec aequatio multiplicata fiat integrabilis.

Solutio.

Ponatur $\partial y = p \partial x$, ut habeatur forma differentialis primi gradus

$$\partial p + A p \partial x + B y \partial x = X \partial x,$$

quae multiplicata per V, functionem quandam ipsius x, fiat integrabilis; scilicet

$$V \partial p + A V p \partial x + B V y \partial x = V X \partial x,$$

ubi cum posterius membrum $V X \partial x$ sit integrabile, idem in priori eveniat, necesse est. At primo perspicuum est ejus integralis partem fore Vp, unde id ponatur $Vp + S$ ut sit $Vp + S = \int V X \partial x$, fietque

$$\partial S = -p\partial V + AVp\partial x + BVy\partial x, \text{ seu}$$
$$\partial S = \partial y\left(AV - \frac{\partial V}{\partial x}\right) + BVy\partial x,$$

quae forma integrabilis reddi potest sumendo $V = e^{\lambda x}$, erit enim

$$\partial S = e^{\lambda x}[(A-\lambda)\partial y + By\partial x] \text{ et } S = (A-\lambda)e^{\lambda x}y,$$

ubi λ ita debet accipi, ut fiat

$$A\lambda - \lambda\lambda = B \text{ seu } \lambda\lambda - A\lambda + B = 0.$$

Tum ergo erit

$$e^{\lambda x}p + (A-\lambda)e^{\lambda x}y = \int e^{\lambda x}X\partial x, \text{ seu}$$
$$\partial y + (A-\lambda)y\partial x = e^{-\lambda x}\partial x\int e^{\lambda x}X\partial x,$$

quae jam per $e^{(A-\lambda)x}$ multiplicata denuo fit integrabilis, datque

$$e^{(A-\lambda)x}y = \int e^{(A-2\lambda)x}\partial x\int e^{\lambda x}X\partial x.$$

Cum λ sit una radix aequationis $\lambda\lambda - A\lambda + B = 0$, si ambas ejus radices ponamus f et g, ut sit $\lambda = f$, erit $A - \lambda = g$, et aequatio integralis

$$e^{gx}y = \int e^{(g-f)x}\partial x\int e^{fx}X\partial x, \text{ seu}$$
$$e^{gx}y = \frac{1}{g-f}e^{(g-f)x}\int e^{fx}X\partial x - \frac{1}{g-f}\int e^{gx}X\partial x,$$

quae abit in formam supra inventam

$$y = \frac{1}{g-f}e^{-fx}\int e^{fx}X\partial x - \frac{1}{g-f}e^{-gx}\int e^{gx}X\partial x.$$

Corollarium 1.

866. Aequatio ergo proposita seu inde nata

$$\partial p + Ap\partial x + By\partial x = X\partial x$$

fit integrabilis si ducatur in $e^{\lambda x}$, existente $\lambda\lambda - A\lambda + B = 0$, sicque duplex habetur factor vel e^{fx} vel e^{gx}.

Corollarium 2.

867. Ea autem per factorem e^{fx} multiplicata, ejus integrale erit

$$\partial y + gy\partial x = e^{-fx}\partial x \int e^{fx} X \partial x,$$

sicque per integrationem ad aequationem differentialem primi gradus reducitur, quae denuo integrabilis redditur si per e^{gx} multiplicetur.

Scholion.

868. Multiplicatorem V ita determinari oportebat, ut formula $\partial y(AV - \frac{\partial V}{\partial x}) + BVy\partial x$ fieret per se integrabilis. Tum autem cum V sit functio ipsius x, integrale erit $y(AV - \frac{\partial V}{\partial x})$, unde fiat necesse est

$$A\partial V - \frac{\partial\partial V}{\partial x} = BV\partial x, \text{ seu } \partial\partial V - A\partial x\partial V + BV\partial x^2 = 0,$$

a cujus aequationis integratione pendet inventio factoris quaesiti V. Sufficit autem ejus integrale particulare sumsisse, dummodo enim aequatio proposita integrabilis reddatur, constans arbitraria pro integrali completo reddendo ipsa integratione introducitur.

Problema 108.

869. Sumto elemento ∂x constante, si proponatur haec aequatio

$$\partial\partial y + P\partial y\partial x + Qy\partial x^2 = X\partial x^2,$$

ubi P, Q et X sint functiones quaecunque ipsius x, invenire multiplicatorem V, qui sit functio ipsius x, quo illa aequatio integrabilis reddatur.

Solutio.

Quia aequatio per V multiplicata

$$V\partial\partial y + VP\partial y\partial x + VQy\partial x^2 = VX\partial x^2,$$

integrabilis existit, prioris partis integrale ponatur $V\partial y + Sy\partial x$, aliam enim formam habere nequit, ac fieri oportet

$$VP\partial y\partial x + VQy\partial x^2 = \partial y\partial V + S\partial y\partial x + y\partial S\partial x,$$

ubi cum S sit necessario functio ipsius x, erit

$$VP\partial x = \partial V + S\partial x \text{ et } VQ\partial x = \partial S.$$

Inde autem est $S = VP - \frac{\partial V}{\partial x}$, quare multiplicator V definiri debet ex hac aequatione

$$VQ\partial x = V\partial P + P\partial V - \frac{\partial\partial V}{\partial x}, \text{ seu}$$

$$\partial\partial V - P\partial V\partial x + V\partial x(Q\partial x - \partial P) = 0,$$

quae ergo si resolvi potuerit, vel si saltem ejus integrale quodpiam particulare innotescat, ut habeatur multiplicator V, aequationis propositae integrale erit

$$V\partial y + y + y(VP\partial x - \partial V) = \partial x\int VX\partial x,$$

quae porro integrabilis redditur, si ducatur in $\frac{1}{VV}e^{\int P\partial x}$, obtinebitur enim integrale

$$\frac{y}{V}e^{\int P\partial x} = \int\frac{\partial x}{VV}e^{\int P\partial x}\int VX\partial x, \text{ seu}$$

$$y = e^{-\int P\partial x}V\int e^{\int P\partial x}\frac{\partial x}{VV}\int VX\partial x,$$

quo duplici signo integrali gemina constans arbitraria introducitur, integrale completum constituens.

Corollarium 1.

870. Inventio ergo multiplicatoris V pendet etiam a resolutione aequationis differentio-differentialis, quae autem proposita simplicior est aestimanda, quod functionem X non involvat, et quantitas V cum suis differentialibus ∂V et $\partial\partial V$ ubique unam dimensionem constituat.

Corollarium 2.

871. Quodsi ergo ponatur $V = e^{\int v\partial x}$, quantitas v determinabitur per hanc aequationem differentialem primi gradus

$$\partial v + vv\partial x - Pv\partial x + Q\partial x - \partial P = 0,$$

cujus si saltem integrale particulare constet, integratio aequationis propositae absolvi poterit.

Corollarium. 3.

872. Dato autem multiplicatore V vicissim ratio aequationis propositae definitur, ut hoc modo integrabilis evadat. Erit enim vel

$$Q = \frac{\partial P}{\partial x} + \frac{P \partial V}{V \partial x} - \frac{\partial \partial V}{V \partial x^2}, \text{ vel}$$

$$\partial P + \frac{P \partial V}{V} = Q \partial x - \frac{\partial \partial V}{V \partial x}, \text{ vel integrando}$$

$$P V = \frac{\partial V}{\partial x} + \int Q V \partial x, \text{ seu } P = \frac{\partial V}{V \partial x} + \frac{\int Q V \partial x}{V}.$$

Exemplum 1.

873. *Definire formam aequationis differentio-differentialis*

$$\partial \partial y + P \partial y \partial x + Q y \partial x^2 = X \partial x^2,$$

ut multiplicata per $e^{\lambda x}$ *integrabilis evadat.*

Posito multiplicatore $V = e^{\int v \partial x} = e^{\lambda x}$, erit $v = \lambda$, et satisfieri oportet huic aequationi

$$\lambda \lambda \partial x - \lambda P \partial x + Q \partial x - \partial P = 0,$$

unde fit $Q = \lambda P - \lambda \lambda + \frac{\partial P}{\partial x}$. Primum ergo hoc evenit si fuerint P et Q constantes, puta $P = A$ et $Q = B$, ac tum λ definiri oportet ex hac aequatione $\lambda \lambda - A \lambda + B = 0$, qui est casus supra tractatus. Praeterea vero qualiscunque functio P fuerit ipsius x, modo sit $Q = \lambda P - \lambda \lambda + \frac{\partial P}{\partial x}$, aequatio in $e^{\lambda x}$ ducta erit integrabilis, integrali existente

$$e^{\lambda x} [\partial y + y \partial x (P - \lambda)] = \partial x \int e^{\lambda x} X \partial x, \text{ seu}$$

$$\partial y + (P - \lambda) y \partial x = e^{-\lambda x} \partial x \int e^{\lambda x} X \partial x,$$

quae ulterius per $e^{\int P \partial x - \lambda x}$ multiplicata et integrata, dat

$$y = e^{-\int P \partial x + \lambda x} \int e^{\int P \partial x - 2 \lambda x} \partial x \int e^{\lambda x} X \partial x.$$

Corollarium.

874. Sit $P = A + \alpha x$ et $Q = B + \beta x$, erit

$$B + \beta x = A\lambda + \alpha\lambda x - \lambda\lambda + \alpha, \text{ ergo}$$

$$B = A\lambda - \lambda\lambda + \alpha \text{ et } \beta = \alpha\lambda,$$

unde ob $\lambda = \frac{\beta}{\alpha}$, coëfficientes A, B, α, β, ita comparatos esse oportet, ut sit

$$B\alpha\alpha = A\alpha\beta - \beta\beta + \alpha^3, \text{ seu } B\alpha\alpha - \beta\beta = \alpha(A\beta + \alpha\alpha).$$

Exemplum 2.

875. *Definire formam aequationis differentio-differentialis*

$$\partial\partial y + P\partial y\partial x + Qy\partial x^2 = X\partial x^2,$$

ut per $e^{\int v\partial x}$, *existente* $v = \frac{\lambda}{x} + \mu x^n$, *multiplicata fiat integrabilis.*

Cum esse debeat

$$\partial v + vv\partial x - Pv\partial x + Q\partial x - \partial P = 0; \text{ erit}$$

$$\left.\begin{aligned} & -\frac{\lambda}{xx} + \mu n x^{n-1} - \frac{\lambda P}{x} - \mu P x^n \\ & + \frac{\lambda\lambda}{xx} + 2\lambda\mu x^{n-1} + \mu\mu x^{2n} + Q - \frac{\partial P}{\partial x} \end{aligned}\right\} = 0,$$

ergo

$$Q = \frac{\lambda(1-\lambda)}{xx} - (2\lambda + n)\mu x^{n-1} - \mu\mu x^{2n} + \frac{\lambda P}{x} + \mu P x^n + \frac{\partial P}{\partial x}.$$

Ponamus $P = \frac{\alpha}{x} + \beta x^n$, erit

$$Q = \frac{1}{xx}(\lambda - \lambda\lambda + \alpha\lambda - \alpha) + x^{n-1}(\beta\lambda + \alpha\mu + \beta n - 2\lambda\mu - n\mu)$$
$$+ x^{2n}(\beta\mu - \mu\mu).$$

Sit $Q = \frac{\gamma}{xx} + \delta x^{n-1} + \varepsilon x^{2n}$, fierique oportet

$$\lambda\lambda - (\alpha + 1)\lambda + \alpha + \gamma = 0,$$

$$\beta(\lambda + n) + \mu(\alpha - 2\lambda - n) = \delta, \text{ et } \mu(\beta - \mu) = \varepsilon,$$

unde non solum pro multiplicatore litterae λ et μ, sed etiam certa relatio inter litteras α, β, γ, δ, ε, definitur.

Veluti si sit $\gamma = 0$ et $\delta = 0$, erit $(\lambda - \alpha)(\lambda - 1) = 0$, unde $\lambda = \alpha$; tum $(\beta - \mu)(\alpha + n) = 0$, ergo $\alpha = \lambda = -n$ et $\mu\mu - \beta\mu + \varepsilon = 0$. Scilicet aequatio

$$\partial\partial y + \partial x \partial y (\beta x^n - \tfrac{n}{x}) + \varepsilon x^{2n} y \partial x^2 = X \partial x^2$$

multiplicatorem recipit $e^{\int v \partial x}$, existente $v = -\frac{n}{x} + \mu x^n$, sumto μ ita ut sit $\mu\mu - \mu + \varepsilon = 0$. Erit ergo multiplicator

$$V = \frac{1}{x^n} e^{\frac{\mu}{n+1} x^{n+1}} \quad \text{et} \quad e^{\int P \partial x} = \frac{1}{x^n} e^{\frac{\beta}{n+1} x^{n+1}}.$$

Quare si ponamus $\frac{1}{n+1} x^{n+1} = t$, erit

$$y = x^n e^{-\beta t} \frac{1}{x^n} e^{\mu t} \int e^{\beta t - 2\mu t} x^n \partial x \int \frac{e^{\mu t} X \partial x}{x^n}, \text{ seu}$$

$$y = e^{(\mu - \beta) t} \int e^{(\beta - 2\mu) t} x^n \partial t \int \frac{e^{\mu t} X \partial x}{x^n}.$$

Corollarium 1.

876. Si sumatur $\gamma = 0$ et $\varepsilon = 0$, erit

$\mu = \beta$, $\beta(\alpha - \lambda) = \delta$ et $(\lambda - \alpha)(\lambda - 1) = 0$, hinc

$\lambda = 1$, et $\delta = (\alpha - 1)\beta$, ideoque

$P = \frac{\alpha}{x} + \beta x^n$, $Q = (\alpha - 1)\beta x^{n-1}$,

et aequationis

$$\partial\partial y + (\tfrac{\alpha}{x} + \beta x^n) \partial x \partial y + (\alpha - 1) \beta x^{n-1} y \partial x^2 = X \partial x^2,$$

multiplicator $V = e^{\int v \partial x}$, existente $v = \frac{1}{x} + \beta x^n$, ita ut sit

$$V = x e^{\frac{\beta}{n+1} x^{n+1}} \quad \text{et} \quad e^{\int P \partial x} = x^\alpha e^{\frac{\beta}{n+1} x^{n+1}}.$$

Corollarium 2.

877. Hoc ergo casu, posito $\frac{1}{n+1} x^{n+1} = t$, erit integrale

$$y = x^{1-\alpha} \int x^{\alpha - 2} e^{-\beta t} \partial x \int e^{\beta t} X x \partial x,$$

quae forma simplicius exhiberi nequit, propterea quod in genere formula $e^{-\beta t} x^{\alpha - 2} \partial x$ integrationem non admittit.

Scholion.

878. Cum igitur inventio multiplicatorum, qui hujusmodi aequationem

$$\partial\partial y + P \partial x \partial y + Q y \partial x^2 = X \partial x^2,$$

integrabilem reddunt, resolutionem hujus aequationis postulet

$$\partial\partial V - P \partial V \partial x + V \partial x (Q \partial x - \partial P) = 0,$$

quae in hac forma continetur

$$\partial\partial y + P \partial x \partial y + Q y \partial x^2 = 0,$$

videndum est, quomodo hanc formam etiam per multiplicatores tractari oporteat. Cujus multiplicator si fingatur V functio quaedam ipsius x, iterum ad praecedentem formam

$$\partial\partial V - P \partial V \partial x + V \partial x (Q \partial x - \partial P) = 0,$$

devenitur, atque si hujus multiplicator statuatur $= U$ functioni ipsius x, hic definietur per hanc aequationem

$$\partial\partial U + P \partial U \partial x + Q U \partial x^2 = 0,$$

ita ut sufficiat alteram harum duarum aequationum resolvisse. Ac supra quidem, ubi $y = u v$ posuimus, ad hanc posteriorem aequationem pervenimus: at mirum non est harum duarum aequationum alteram ab altera pendere, cum prior ex posteriori nascatur ponendo $U = e^{-\int P \partial x} V$, posterior vero ex priori ponendo $V = e^{\int P \partial x} U$, uti tentanti facile patebit. Quoniam igitur hoc modo difficultatem, si quae occurrit, tollere non licet, investigandum est, an forte ejus-

modi multiplicator, qui utramque variabilem x et y cum suis differentialibus ∂x et ∂y seu $p = \frac{\partial y}{\partial x}$ involvat, negotium conficiat. At vero facile perspicitur exclusis differentialibus hoc fieri non posse; nam si multiplicator esset V functio ipsarum x et y, ex primo termino $\partial\partial y$ nascetur integralis pars $V\partial y$, quae autem differentiata ponendo $\partial V = M\partial x + N\partial y$ involveret in differentiale partem $N\partial y^2$, in aequatione non occurrentem, quae etiam per reliquas integralis partes tolli non posset. Quare rem tentemus ejusmodi multiplicatoribus, qui etiam rationem differentialium $p = \frac{\partial y}{\partial x}$ complectantur; et cum ipsius y cum suis differentialibus ubique sit idem dimensionum numerus, eadem proprietas etiam in multiplicatore insit necesse est; si enim diversae inessent, singulae seorsim negotium essent confecturae.

Problema 109.

879. Sumto elemento ∂x constante, definire conditiones, ut multiplicator hujus formae $Mp + Ny$, existente $p = \frac{\partial y}{\partial x}$ et M et N functionibus ipsius x, integrabilem reddat hanc aequationem

$$\partial\partial y + P\partial x\partial y + Qy\partial x^2 = 0,$$

ubi P et Q sunt functiones ipsius x.

Solutio.

Ob $\partial y = p\partial x$, nostra aequatio est

$$\partial p + Pp\partial x + Qy\partial x = 0,$$

quae per $Mp + Ny$ multiplicata fit

$$\left.\begin{matrix} Mp\partial p + Ny\partial p + MPp\partial y + NPy\partial y + NQyy\partial x \\ + MQy\partial y \end{matrix}\right\} = 0,$$

quam integrabilem esse oportet. Ob terminos differentiali ∂p affectos pars integrabilis erit $\frac{1}{2}Mpp + Nyp$, unde integrale ipsum sta-

tuatur $= \frac{1}{2} Mpp + Nyp + S$. Cujus differentiale cum ipsam illam aequationem praebere debeat, habebimus

$$\begin{aligned}\partial S = MPp\partial y + NPy\partial y + NQyy\partial x \\ + MQy\partial y \\ - \tfrac{1}{2} pp\partial M - yp\partial N \\ - Np\partial y\end{aligned}$$

quam ergo formulam integrabilem esset oportet, quae cum tantum differentialia primi ordinis ∂x et ∂y complectatur, necesse est ut quantitas p ex calculo egrediatur. Posito ergo $\partial M = M'\partial x$ et $\partial N = N'\partial x$, ob $p\partial x = \partial y$, primus terminus continens p ad nihilum redigi debet, ut sit

$$MPp\partial y - \tfrac{1}{2} M'p\partial y - Np\partial y = 0, \text{ seu}$$
$$MP - \tfrac{1}{2} M' - N = 0, \text{ vel } N = MP - \frac{\partial M}{2\partial x}.$$

Tum vero erit

$$\partial S = y\partial y\,(NP + MQ - N') + NQyy\partial x,$$

cujus formulae integrale est

$$S = \tfrac{1}{2} yy\,(NP + MQ - N'), \text{ vel } S = yy\textstyle\int NQ\partial x,$$

quas duas formas congruere oportet, unde fit

$$NP + MQ - \frac{\partial N}{\partial x} = 2\textstyle\int NQ\partial x, \text{ seu}$$
$$N\partial P + P\partial N + M\partial Q + Q\partial M - \frac{\partial\partial N}{\partial x} - 2NQ\partial x = 0,$$

quae aequatio cum illa $N = MP - \frac{\partial M}{2\partial x}$ juncta, conditiones quaesitas determinat, proditque tum aequatio integralis

$$\tfrac{1}{2} Mpp + Nyp + \tfrac{1}{2} yy\left(NP + MQ - \frac{\partial N}{\partial x}\right) = C.$$

Corollarium 1.

880. Si functiones P et Q dentur, indeque M et N definiri oporteat, ob $N = MP - \frac{\partial M}{2\partial x}$, erit

$$\partial N = M\partial P + P\partial M - \frac{\partial\partial M}{2\partial x},$$

et functio M definietur per hanc aequationem

$$\frac{\partial^3 M}{2\partial x^2} - \frac{3P\partial\partial M}{2\partial x}$$

$$+ (PP - \frac{5\partial P}{2\partial x} + 2Q)\partial M + M(2P\partial P - \frac{\partial\partial P}{\partial x} - 2PQ\partial x + \partial Q) = 0,$$

quae ob differentialia tertii ordinis parum juvat.

Corollarium 2.

881. Sin autem multiplicator $Mp + Ny$ detur, ipsa aequatio ita definitur, ut sit primo $P = \frac{N}{M} + \frac{\partial M}{2M\partial x}$, unde ex altera, quae est

$$\partial Q + \frac{Q\partial M}{M} - \frac{2NQ\partial x}{M} = \frac{\partial\partial N}{M\partial x} - \frac{\partial . PN}{M},$$

haecque per $M e^{-2\int\frac{N\partial x}{M}}$ multiplicata, integrale dat

$$MQe^{-2\int\frac{N\partial x}{M}} = \int e^{-2\int\frac{N\partial x}{M}}(\frac{\partial\partial N}{\partial x} - \partial . PN).$$

Corollarium 3.

882. Sit hoc integrale $= Z$ eritque

$$Z = e^{-2\int\frac{N\partial x}{M}}(\frac{\partial N}{\partial x} - PN) + \int e^{-2\int\frac{N\partial x}{M}}(\frac{2N\partial N}{M} - \frac{2PNN\partial x}{M}),$$

quod posterius membrum, pro P valore substituto, abit in

$$\int e^{-2\int\frac{N\partial x}{M}}(\frac{2N\partial N}{M} - \frac{2N^3\partial x}{MM} - \frac{NN\partial M}{MM}),$$

cujus integrale est manifesto $e^{-2\int\frac{N\partial x}{M}}\frac{NN}{M}$, ita ut sit

$$Z = e^{-2\int\frac{N\partial x}{M}}(\frac{\partial N}{\partial x} - \frac{N\partial M}{2M\partial x}) + C, \text{ ideoque}$$

$$Q = \frac{C}{M}e^{2\int\frac{N\partial x}{M}} + \frac{\partial N}{M\partial x} - \frac{N\partial M}{2MM\partial x}.$$

Corollarium 4.

883. Proposita ergo hac aequatione

$$\frac{\partial\partial y}{\partial x}+\left(\frac{N}{M}+\frac{\partial M}{2M\partial x}\right)\partial y+y\left(\frac{C\partial x}{M}e^{2\int\frac{N\partial x}{M}}+\frac{\partial N}{M}-\frac{N\partial M}{2MM}\right)=0,$$

eam per $\frac{M\partial y}{\partial x}+Ny$ multiplicando, integrale fit

$$\frac{M\partial y^2}{2\partial x^2}+\frac{Ny\partial y}{\partial x}+\tfrac{1}{2}yy\left(Ce^{2\int\frac{N\partial x}{M}}+\frac{NN}{M}\right)=\text{Const.}$$

Scholion.

884. Cum ergo pro M et N quascunque functiones ipsius x accipere liceat, innumerabiles hinc nacti sumus aequationum differentio-differentialium formas, quas ope multiplicatoris $\frac{M\partial y}{\partial x}+Ny$ integrare possumus. Forma scilicet generalis, quae hoc multiplicatore integrabilis redditur, est ut vidimus

$$\frac{\partial\partial y}{\partial x}+\frac{\partial y}{2M\partial x}(\partial M+2N\partial x)$$
$$+\frac{y}{2MM}\left(2M\partial N-N\partial M+2CMe^{2\int\frac{N\partial x}{M}}\partial x\right),$$

ipso integrali existente

$$\frac{M\partial y^2}{2\partial x^2}+\frac{Ny\partial y}{\partial x}+\tfrac{1}{2}yy\left(\frac{NN}{M}+Ce^{2\int\frac{N\partial x}{M}}\right),$$

ubi perspicuum est partem exponentialem constanti C affectam utrinque omitti posse, cum ea sola ista proprietate sit praedita. Quodsi partem exponentialem ad algebraicam reduceamus ponendo $e^{2\int\frac{N\partial x}{M}}=L$, erit $\frac{2N\partial x}{M}=\frac{\partial L}{L}$ et $N=\frac{M\partial L}{2L\partial x}$, hincque $\partial N=\frac{M\partial\partial L}{2L\partial x}+\frac{\partial L\partial M}{2L\partial x}-\frac{M\partial L^2}{2LL\partial x}$; unde ista forma

$$\frac{\partial\partial y}{\partial x}+\frac{\partial y}{2\partial x}\left(\frac{\partial M}{M}+\frac{\partial L}{L}\right)+\tfrac{1}{2}y\left(\frac{\partial\partial L}{L\partial x}+\frac{\partial L\partial M}{2LM\partial x}-\frac{\partial L^2}{LL\partial x}+\frac{2CL\partial x}{M}\right),$$

quae per $\frac{M\partial y}{\partial x}+\frac{My\partial L}{2L\partial x}$ multiplicata, integrale praebet

$$\frac{M\partial y^2}{2\partial x^2}+\frac{My\partial L\partial y}{2L\partial x^2}+\tfrac{1}{2}yy\left(\frac{M\partial L^2}{4LL\partial x^2}+CL\right).$$

Vel si ponamus $\frac{\partial M}{M} + \frac{\partial L}{L} = \frac{2\partial K}{K}$, ut sit $M = \frac{KK}{L}$, erit nostra aequatio differentio-differentialis

$$\frac{\partial\partial y}{\partial x} + \frac{\partial y}{\partial x}\cdot\frac{\partial K}{K} + \tfrac{1}{2}y\left(\partial\cdot\frac{\partial L}{L\partial x} - \frac{\partial L^2}{2LL\partial x} + \frac{\partial K\partial L}{KL\partial x} + \frac{2CLL\partial x}{KK}\right) = 0,$$

quae per $\frac{KK}{L}\left(\frac{\partial y}{\partial x} + \frac{y\partial L}{2L\partial x}\right)$ multiplicata, dabit integrale

$$\frac{KK}{2L}\left[\frac{\partial y^2}{\partial x^2} + \frac{y\partial L\partial y}{L\partial x^2} + yy\left(\frac{\partial L^2}{4LL\partial x^2} + \frac{CLL}{KK}\right)\right] = \text{Const.}$$

Exemplum 1.

885. Sit $K = x^m(a+x)^n$ et $L = x^\mu(a+x)^\nu$, erit

$$\frac{\partial K}{K\partial x} = \frac{m}{x} + \frac{n}{a+x} = \frac{ma + (m+n)x}{x(a+x)}, \text{ et}$$

$\frac{\partial L}{L\partial x} = \frac{\mu}{x} + \frac{\nu}{a+x}$; unde coëfficiens ipsius $\frac{1}{2}y\partial x$ erit

$$-\frac{\mu}{xx} - \frac{\nu}{(a+x)^2} - \frac{\mu\mu}{2xx} - \frac{\mu\nu}{x(a+x)} - \frac{\nu\nu}{2(a+x)^2} + \frac{m\mu}{xx} + \frac{m\nu+n\mu}{x(a+x)}$$
$$+ \frac{n\nu}{(a+x)^2} + 2Cx^{2\mu-2m}(a+x)^{2\nu-2n}, \text{ seu}$$

$$\frac{\mu(2m-\mu-2)}{2xx} + \frac{m\nu+n\mu-\mu\nu}{x(a+x)} + \frac{\nu(2n-\nu-2)}{2(a+x)^2}$$
$$+ 2Cx^{2\mu-2m}(a+x)^{2\nu-2n},$$

ubi sequentes casus notasse juvabit.

I. Sit $m = \mu + 1$ et $n = \nu$, erit ipsius $\frac{1}{2}y\partial x$ coëfficiens

$$\frac{\mu\mu+4C}{2xx} + \frac{\nu(\mu+1)}{x(a+x)} + \frac{\nu(\nu-2)}{2(a+x)^2}.$$

Hinc ista aequatio

$$\frac{\partial\partial y}{\partial x} + \partial y\left(\frac{\mu+1}{x} + \frac{\nu}{a+x}\right)$$
$$+ \tfrac{1}{4}y\partial x\left[\frac{\mu\mu+4C}{xx} + \frac{2\nu(\mu+1)}{x(a+x)} + \frac{\nu(\nu-2)}{(a+x)^2}\right] = 0$$

multiplicata per

$$x^{\mu+2}(a+x)^\nu\left[\frac{\partial y}{\partial x} + \left(\frac{\mu}{x} + \frac{\nu}{a+x}\right)\frac{y}{2}\right],$$

integrale dat

$$\tfrac{1}{2} x^{\mu+2} (a+x)^{\nu} \left[\frac{\partial y^2}{\partial x^2} + \frac{y \partial y}{\partial x}\left(\frac{\mu}{x} + \frac{\nu}{a+x}\right) + \tfrac{1}{4} y y \left(\frac{\mu\mu+4C}{xx} + \frac{2\mu\nu}{x(a+x)} + \frac{\nu\nu}{(a+x)^2}\right)\right] = \text{Const.}$$

II. Sit $m = \mu + \frac{1}{2}$ et $n = \nu + \frac{1}{2}$, erit ipsius $\frac{1}{2} y \partial x$ coëfficiens

$$\frac{\mu(\mu-1)}{2xx} + \frac{2\mu\nu+\mu+\nu+4C}{2x(a+x)} + \frac{\nu(\nu-1)}{2(a+x)^2}.$$

Hinc ista aequatio

$$\frac{\partial\partial y}{\partial x} + \partial y\left(\frac{2\mu+1}{2x} + \frac{2\nu+1}{2(a+x)}\right) + \tfrac{1}{4} y \partial x \left(\frac{\mu(\pi-1)}{xx} + \frac{2\mu\nu+\mu+\nu+4C}{x(a+x)} + \frac{\nu(\nu-1)}{(a+x)^2}\right) = 0,$$

multiplicata per

$$x^{\mu+1} (a+x)^{\nu+1} \left[\frac{\partial y}{\partial x} + \tfrac{1}{2} y \left(\frac{\mu}{x} + \frac{\nu}{a+x}\right)\right],$$

integrale dabit

$$\tfrac{1}{2} x^{\mu+1} (a+x)^{\nu+1} \left[\frac{\partial y^2}{\partial x^2} + \frac{y \partial y}{\partial x}\left(\frac{\mu}{x} + \frac{\nu}{a+x}\right) + \tfrac{1}{4} y y \left(\frac{\mu\mu}{xx} + \frac{2\mu\nu+4C}{x(a+x)} + \frac{\nu\nu}{(a+x)^2}\right)\right] = \text{Const.}$$

III. Sit $m = \mu$ et $n = \nu + 1$, erit ipsius $\frac{1}{2} y \partial x$ coëfficiens

$$\frac{\mu(\mu-2)}{2xx} + \frac{\mu(\nu+1)}{x(a+x)} + \frac{\nu\nu+4C}{2(a+x)^2}.$$

Hinc ista aequatio

$$\frac{\partial\partial y}{\partial x} + \partial y\left(\frac{\mu}{x} + \frac{\nu+1}{a+x}\right) + \tfrac{1}{4} y \partial x \left(\frac{\mu(\mu-2)}{xx} + \frac{2\mu(\nu+1)}{x(a+x)} + \frac{\nu\nu+4C}{(a+x)^2}\right) = 0,$$

multiplicata per

$$x^{\mu} (a+x)^{\nu+2} \left[\frac{\partial y}{\partial x} + \tfrac{1}{2} y \left(\frac{\mu}{x} + \frac{\nu}{a+x}\right)\right],$$

dabit integrale

$$\tfrac{1}{2} x^{\mu} (a+x)^{\nu+2} \left[\frac{\partial y^2}{\partial x^2} + \frac{y \partial y}{\partial x}\left(\frac{\mu}{x} + \frac{\nu}{a+x}\right) + \tfrac{1}{4} y y \left(\frac{\mu\mu}{xx} + \frac{2\mu\nu}{x(a+x)} + \frac{\nu\nu+4C}{(a+x)^2}\right)\right] = \text{Const.}$$

Corollarium 1.

886. Sit casu primo $\nu = 2$, $C = -\frac{1}{4}\mu\mu$, habebitur haec aequatio

$$\frac{\partial\partial y}{\partial x} + \frac{(\mu+1)a+(\mu+3)x}{x(a+x)}\partial y + \frac{(\mu+1)y\partial x}{x(a+x)} = 0,$$

quae per

$$x^{\mu+2}(a+x)^2\left(\frac{\partial y}{\partial x} + \frac{\mu a+(\mu+2)x}{2x(a+x)}y\right)$$

multiplicata, praebet integrale

$$\tfrac{1}{2}x^{\mu+2}(a+x)^2\left[\frac{\partial y^2}{\partial x^2} + \frac{\mu a+(\mu+2)x}{x(a+x)}\cdot\frac{y\partial y}{\partial x} + yy\left(\frac{\mu}{x(a+x)} + \frac{1}{(a+x)^2}\right)\right] = \text{Const.}$$

Corollarium 2.

887. Sit casu tertio $\mu = 2$ et $4C = -\nu\nu$, habebitur ista aequatio

$$\frac{\partial\partial y}{\partial x} + \partial y\cdot\frac{2a+(\nu+3)x}{x(a+x)} + \frac{(\nu+1)y\partial x}{x(a+x)} = 0,$$

quae multiplicata per $xx(a+x)^{\nu+2}\left[\frac{\partial y}{\partial x} + \frac{1}{2}y\left(\frac{2}{x} + \frac{\nu}{a+x}\right)\right]$, dabit integrale

$$\tfrac{1}{2}xx(a+x)^{\nu+2}\left[\frac{\partial y^2}{\partial x^2} + \frac{y\partial y}{\partial x}\left(\frac{2}{x} + \frac{\nu}{a+x}\right) + yy\left(\frac{1}{xx} + \frac{\nu}{x(a+x)}\right)\right] = \text{Const.}$$

Exemplum 2.

888. Sit $K = x^m(aa+xx)^n$ et $L = x^\mu(aa+xx)^\nu$, erit $\frac{\partial K}{K\partial x} = \frac{m}{x} + \frac{2nx}{aa+xx}$ et $\frac{\partial L}{L\partial x} = \frac{\mu}{x} + \frac{2\nu x}{aa+xx}$,
et aequatio differentio-differentialis hanc induet formam

$$\frac{\partial\partial y}{\partial x} + \partial y\left(\frac{m}{x} + \frac{2nx}{aa+xx}\right) + \tfrac{1}{2}y\partial x\left\{\begin{array}{l}\frac{\mu(2m-\mu-2)}{2xx} + \frac{2x\mu+2\nu(m-\mu+1)}{aa+xx} + \frac{2\nu(2n-\nu-2)xx}{(aa+xx)^2}\\ + \frac{2Cx^{2\mu-2m}}{(aa+xx)^{2n-2\nu}}\end{array}\right\} = 0,$$

cujus in $x^{2m-\mu}(aa+xx)^{2n-\nu}\left[\frac{\partial y}{\partial x} + \frac{1}{2}y\left(\frac{\mu}{x} + \frac{2\nu x}{aa+xx}\right)\right]$ ductae integrale erit

$$\tfrac{1}{2}x^{2m-\mu}(aa+xx)^{2n-\nu}\times\left\{\begin{array}{l}\frac{\partial y^2}{\partial x^2} + \frac{y\partial y}{\partial x}\left(\frac{\mu}{x} + \frac{2\nu x}{aa+xx}\right)\\ + \frac{1}{4}yy\left[\left(\frac{\mu}{x} + \frac{2\nu x}{aa+xx}\right)^2 + \frac{4Cx^{2\mu-2m}}{(aa+xx)^{2n-2\nu}}\right]\end{array}\right\} = \text{Const.}$$

Evolvamus hic casus, quibus aequatio differentio-differentialis hanc

obtinet formam

$$\frac{\partial\partial y}{\partial x}+\partial y\left(\frac{m}{x}+\frac{2nx}{aa+xx}\right)+\tfrac{1}{2}y\partial x\left(D+\frac{E}{xx}+\frac{F}{aa+xx}+\frac{Gxx}{(aa+xx)^2}\right)=0.$$

I. Sumatur $\mu = m$ et $\nu = n$,

eritque $D = 2C$, $E = \frac{1}{2}m(m-2)$, $F = 2n(m+1)$, et

$$G = 2n(n-2).$$

II. Sumatur $\mu = m - 1$ et $\nu = n$,

eritque $D = 0$, $E = 2C + \frac{1}{2}(m-1)^2$, $F = 2n(m+1)$, et

$$G = 2n(n-2).$$

III. Sumatur $\mu = m - 1$ et $2n - 2\nu = -1$, seu $\nu = n + \frac{1}{2}$, erit ultimus terminus $\frac{2C(aa+xx)}{xx} = 2C + \frac{2Caa}{xx}$. Ergo

$$D = 2C,\ E = 2Caa + \tfrac{1}{2}(m-1)^2,\ E = 2(mn+n+1),$$

$$G = \tfrac{1}{2}(2n+1)(2n-5).$$

IV. Sumatur $\mu = m$ et $2n - 2\nu = 1$, seu $\nu = n - \frac{1}{2}$, erit ultimus terminus $\frac{2C}{aa+xx}$, ideoque

$$D = 0,\ E = \tfrac{1}{2}m(m-2),\ F = 2C + 2mn + 2n - 1,$$

$$G = \tfrac{1}{2}(2n-1)(2n-3).$$

V. Sumatur $\mu = m + 1$ et $\nu = n - \frac{1}{2}$, erit ultimus terminus $-\frac{2Cxx}{aa+xx} = 2C - \frac{2Caa}{aa+xx}$, ideoque

$$D = 2C,\ E = \tfrac{1}{2}(m+1)(m-2),\ F = -2Caa + 2n(m+1),$$

$$G = \tfrac{1}{2}(2n-1)(2n-3).$$

VI. Sit $\mu = m - 1$ et $\nu = n - \frac{1}{2}$, erit ultimus terminus $\frac{2C}{xx(aa+xx)} = \frac{2C}{aaxx} - \frac{2C}{aa(aa+xx)}$, unde fit

$$D = 0,\ E = \frac{2C}{aa} + \tfrac{1}{2}(m-1)^2,\ F = \frac{-2C}{aa} + 2mn + 2n - 2,$$

$$G = \tfrac{1}{2}(2n-1)(2n-3)$$

VII. Sit $\mu = m + 1$ et $2n - 2\nu = 2$, seu $\nu = n - 1$, erit

terminus ultimus $\frac{2Cxx}{(aa+xx)^2}$, ideoque

$$D=0,\ E=\tfrac{1}{2}(m+1)(m-3),\ F=2n(m+1),$$
$$G=2C+2(n-1)^2.$$

VIII. Sit $\mu=m+2$ et $\nu=n-1$, erit terminus ultimus $\frac{2Cx^4}{(aa+xx)^2}=2C-\frac{2Caa}{aa+xx}-\frac{2Caaxx}{(aa+xx)^2}$, hincque

$$D=2C,\ E=\tfrac{1}{2}(m+2)(m-4),\ F=-2Caa+2mn+2n+2,$$
$$\text{et } G=-2Caa+2(n-1)^2.$$

IX. Sit $\mu=m$ et $\nu=n-1$, erit terminus ultimus

$$\frac{2C}{(aa+xx)^2}=\frac{2C}{aa(aa+xx)}-\frac{2Cxx}{a^2(aa+xx)^2},$$

hincque

$$D=0,\ E=\tfrac{1}{2}m(m-2),\ F=\frac{2C}{aa}+2mn+2n-2,$$
$$G=\frac{-2C}{aa}+2(n-1)^2.$$

X. Sit $\mu=m-1$ et $\nu=n-1$, erit terminus ultimus

$$\frac{2C}{xx(aa+xx)^2}=\frac{2C}{a^4xx}-\frac{4C}{a^4(aa+xx)}+\frac{2Cxx}{a^4(aa+xx)^2},$$

hincque

$$D=0,\ E=\frac{2C}{a^4}+\tfrac{1}{2}(m-1)^2,\ F=\frac{-4C}{a^4}+2mn+2n-4,$$
$$G=\frac{2C}{a^4}+2(n-1)^2.$$

Problema 110.

889. Sumto elemento ∂x constante, si K et L denotent functiones quascunque ipsius x, invenire integrale completum hujus aequationis differentio-differentialis

$$\frac{\partial\partial y}{\partial x}+\frac{\partial y}{\partial x}\cdot\frac{\partial K}{K}+\tfrac{1}{2}y\left(\partial\cdot\frac{\partial L}{L\partial x}-\frac{\partial L^2}{2LL\partial x}+\frac{\partial K\partial L}{KL\partial x}+\frac{2CLL\partial x}{KK}\right)=0.$$

Solutio.

Quoniam haec aequatio integrabilis redditur, si multiplicetur per $\frac{KK}{L}\left(\frac{\partial y}{\partial x}+\frac{y\partial L}{2L\partial x}\right)$, ejus integrale completum, ut supra vidimus, est

$$\frac{KK}{2L}\left[\left(\frac{\partial y}{\partial x}+\frac{y\partial L}{2L\partial x}\right)^2+\frac{CLL}{KK}yy\right]=\text{Const.}$$

quam aequationem differentialem primi gradus adhuc integrari oportet; quod cum ob constantem indefinitam maxime sit difficile, ea neglecta, primo saltem integrale particulare investigemus. Erit ergo ex aequatione

$$\left(\frac{\partial y}{\partial x}+\frac{y\partial L}{2L\partial x}\right)^2+\frac{CLL}{KK}yy=0$$

radicem extrahendo

$$\frac{\partial y}{\partial x}+\frac{y\partial L}{2L\partial x}=\frac{Ly}{K}\sqrt{-C}, \text{ seu } \frac{\partial y}{y}+\frac{\partial L}{2L}=\frac{L\partial x}{K}\sqrt{-C},$$

unde fit

$$y\sqrt{L}=a e^{\int\frac{L\partial x}{K}\sqrt{-C}}.$$

Cum igitur aequationi differentio-differentiali propositae satisfaciant hi duo valores

$$y=\frac{\alpha}{\sqrt{L}}e^{\int\frac{L\partial x}{K}\sqrt{-C}} \text{ et } y=\frac{\beta}{\sqrt{L}}e^{-\int\frac{L\partial x}{K}\sqrt{-C}},$$

bini conjuncti etiam satisfacient, quibus quoniam duae constantes arbitrariae introducuntur, ejus integrale completum erit

$$y=\frac{\alpha}{\sqrt{L}}e^{\int\frac{L\partial x}{K}\sqrt{-C}}+\frac{\beta}{\sqrt{L}}e^{-\int\frac{L\partial x}{K}\sqrt{-C}},$$

quae expressio valet, si $\sqrt{-C}$ fuerit quantitas realis, sin autem sit imaginaria, erit

$$y=\frac{\gamma}{\sqrt{L}}\sin.\left(\int\frac{L\partial x}{K}\sqrt{C}+\zeta\right):$$

sicque habetur integrale completum aequationis differentio-differentialis propositae.

Corollarium 1.

890. Hinc igitur aequationis differentialis primi gradus:

$$\left(\frac{\partial y}{\partial x}+\frac{y\partial L}{2L\partial x}\right)^2+\frac{CLLy^2}{KK}=\frac{AL}{KK},$$

quae per se satis est difficilis, integrale assignare valemus, quod est

$$y = \frac{\alpha}{\sqrt{L}} e^{\int \frac{L\partial x}{K} \sqrt{-C}} + \frac{\beta}{\sqrt{L}} e^{-\int \frac{L\partial x}{K} \sqrt{-C}},$$

si modo debita relatio constantium α et β respectu constantis A definiatur.

Corollarium 2.

891. Erit autem per $\sqrt{L}$ multiplicando et differentiando

$$\partial y \sqrt{L} + \frac{y\partial L}{2\sqrt{L}} = \frac{\alpha L\partial x}{K}\sqrt{-C}\,.\,e^{\int \frac{L\partial x}{K}\sqrt{-C}}$$
$$- \frac{\beta L\partial x}{K}\sqrt{-C}\,.\,e^{-\int \frac{L\partial x}{K}\sqrt{-C}}.$$

Hinc

$$\frac{\partial y}{\partial x} + \frac{y\partial L}{2L\partial x} = \frac{\sqrt{-C}\,L}{K}\left(\alpha e^{\int \frac{L\partial x}{K}\sqrt{-C}} - \beta e^{-\int \frac{L\partial x}{K}\sqrt{-C}}\right),$$

unde fit

$$\frac{AL}{KK} = \frac{-CL}{KK}\left(\alpha e^{\int \frac{L\partial x}{KK}\sqrt{-C}} - \beta e^{-\int \frac{L\partial x}{K}\sqrt{-C}}\right)^2$$
$$+ \frac{CL}{KK}\left(\alpha e^{\int \frac{L\partial x}{K}\sqrt{-C}} + \beta e^{-\int \frac{L\partial x}{K}\sqrt{-C}}\right)^2,$$

ideoque $A = 4C\alpha\beta$ seu $\beta = \frac{A}{4C\alpha}$.

Scholion 1.

892. Quamvis ergo aequationem propositam ope idonei multiplicatoris integrare licuerit, altera tamen integratio maximis difficultatibus premi videbatur. Interim tamen ope substitutionis aequatio illa differentialis primi gradus tractatu facilis redditur; posito enim $y = \frac{z}{\sqrt{L}}$, ut sit $\partial y \sqrt{L} + \frac{y\partial L}{2\sqrt{L}} = \partial z$, oritur $\left(\frac{\partial z}{\partial x \sqrt{L}}\right)^2 + \frac{CLzz}{KK} = \frac{AL}{KK}$, hinc $\frac{\partial z}{\partial x} = \frac{L}{K}\sqrt{(A - Czz)}$, seu $\frac{\partial z}{\sqrt{(A-Czz)}} = \frac{L\partial x}{K}$, quae integrata dat

$$l[z\sqrt{-C}+\sqrt{(A-Czz)}]=\int\frac{L\partial x}{K}\sqrt{-C}+lB,$$

unde praecedens integrale eruitur. Caeterum forma nostrae aequationis differentio-differentialis aliquanto commodius exhiberi potest hoc modo: Si P et R sint functiones quaecunque ipsius x, sumaturque elementum ∂x constans, hujus aequationis

$$\partial\partial y-\partial y\left(\frac{\partial P}{P}+\frac{\partial R}{R}\right)-y\left(\partial.\frac{\partial P}{P}-\frac{\partial P\partial R}{PR}+\frac{aaRR\partial x^2}{PP}\right)=0,$$

bis integratae integrale completum est

$$y=\alpha P e^{a\int\frac{R\partial x}{P}}+\beta P e^{-a\int\frac{R\partial x}{P}},$$

siquidem a sit quantitas realis. At si $a=0$, erit

$$y=P\left(\alpha+\beta\int\frac{R\partial x}{P}\right).$$

Sin autem sit $aa=-cc$, erit

$$y=\alpha P\sin.\left(\beta+c\int\frac{R\partial x}{P}\right).$$

Tum vero illa aequatio integrabilis redditur, si multiplicetur per $\frac{1}{RR\partial x^2}\left(\partial y-\frac{y\partial P}{P}\right)$, eritque integrale primum

$$\frac{1}{2RR\partial x^2}\left[\left(\partial y-\frac{y\partial P}{P}\right)^2-\frac{aaRR}{PP}y^2\partial x^2\right]=\text{Const.}$$

Hinc patet in illa aequatione differentio-differentiali commode hanc substitutionem adhiberi $y=Pz$, qua ea transformatur in

$$\partial\partial z+\partial z\left(\frac{\partial P}{P}-\frac{\partial R}{R}\right)-\frac{aaRR}{PP}z\partial x^2=0,$$

quae per $\frac{PP\partial z}{RR\partial x^2}$ multiplicata, sponte fit integrabilis. Quin etiam posito $\frac{P}{R}=S$, ut habeatur

$$\partial\partial z+\frac{\partial S\partial z}{S}-\frac{aaz\partial x^2}{SS}=0,$$

multiplicator $\frac{SS\partial z}{\partial x^2}$, statim dat integrale

$$\frac{SS\partial z^2}{2\partial x^2}-\tfrac{1}{2}aazz=\text{Const.}$$

Scholion 2.

893. Vicissim ergo ex hac forma simplicissima

$$SS\partial\partial z + S\partial S\partial z - aaz\partial x^2 = 0,$$

quae per ∂z multiplicata integrabilis redditur, formas magis complicatas derivare potuissemus, ponendo $z = \frac{y}{P}$ et $S = \frac{P}{R}$. Quae quamquam in formis generalibus satis perspicua, tamen in exemplis determinatis plerumque haec derivatio nimis est occulta, quam ut menti occurrere possit. Veluti in casibus §. 888. evolutis, si N°. IX. sumamus $m = 2$, et $C = (n-1)^2 aa$, fiet $D = 0$, $E = 0$, $F = 2n(n+1)$, et $G = 0$, unde habetur haec aequatio

$$\frac{\partial\partial y}{\partial x} + 2\partial y\left(\frac{1}{x} + \frac{nx}{aa+xx}\right) + \frac{n(n+1)y\partial x}{aa+xx} = 0, \text{ seu}$$

$$\partial\partial y + \frac{2\partial x\partial y[aa+(n+1)xx]}{x(aa+xx)} + \frac{n(n+1)y\partial x^2}{aa+xx} = 0,$$

quae integrabilis redditur ope multiplicatoris

$$xx(aa+xx)^{n+1}\left(\frac{\partial y}{\partial x} + \frac{y(aa+nxx)}{x(aa+xx)}\right),$$

integrali existente

$$\tfrac{1}{2}xx(aa+xx)^{n+1}\left[\left(\frac{\partial y}{\partial x} + \frac{y(aa+nxx)}{x(aa+xx)}\right)^2 + \frac{(n-1)^2aayy}{(aa+xx)^2}\right] = \text{Const.}$$

Pro integrali ergo particulari erit

$$\frac{\partial y}{y} + \frac{\partial x}{x} + \frac{(n-1)x\partial x}{aa+xx} = \pm\frac{(n-1)a\partial x\sqrt{-1}}{aa+xx},$$

unde colligitur

$$xy(aa+xx)^{\frac{n-1}{2}} = \alpha\left(\frac{a+x\sqrt{-1}}{a-x\sqrt{-1}}\right)^{\pm\frac{(n-1)}{2}}.$$

Ergo bina integralia particularia conjuncta dant

$$y = \frac{\alpha}{x}(a-x\sqrt{-1})^{-n+1} + \frac{\beta}{x}(a+x\sqrt{-1})^{-n+1},$$

integrale completum. Hoc autem casu aequatio nostra ad formam simplicissimam reducetur ope substitutionis

$$y = \frac{z}{x}(aa + xx)^{\frac{1-n}{2}},$$

cujus ratio et inventio difficilius perspicitur.

Problema 111.

894. Sumto elemento ∂x constante, investigare conditiones quibus aequatio differentio-differentialis

$$\partial\partial y + P\partial x\partial y + Q y\partial x^2 = 0,$$

integrabilis redditur, ope multiplicatoris hujus formae

$$\frac{y\partial x^2}{L\partial y^2 + My\partial y\partial x + Nyy\partial x^2},$$

denotantibus litteris L, M, N, P, Q functiones ipsius x.

Solutio.

Tribuatur denominatori hujus fractionis talis forma

$$(\partial y + Ry\partial x)(\partial y + Sy\partial x),$$

ac levi attentione adhibita patet, integrale hujusmodi formam esse habiturum

$$V + l\frac{\partial y + Ry\partial x}{\partial y + Sy\partial x} = \text{Const.}$$

cujus ergo differentiale aequationem propositam producere debet. Dat autem differentiatio

$$\partial V + \frac{(S-R)y\partial x\partial\partial y + (R-S)\partial x\partial y^2 + y\partial x\partial y(\partial R - \partial S) + yy\partial x^2(S\partial R - R\partial S)}{(\partial y + Ry\partial x)(\partial y + Sy\partial x)} = 0,$$

quae ad communem denominatorem reducta, abit in

$$\left.\begin{aligned}(S-R)y\partial x\partial\partial y + (R-S)\partial x\partial y^2 + y\partial x\partial y(\partial R - \partial S) + yy\partial x^2(S\partial R - R\partial S)\\ + \partial V\partial y^2 + (R+S)y\partial x\partial y\partial V + RSyy\partial x^2\partial V\end{aligned}\right\} = 0.$$

Statuatur $\partial V = (S-R)\partial x$, ut aequatio per y divisibilis evadat, sicque orietur haec aequatio

$$\left.\begin{aligned}(S-R)\partial\partial y + \partial y(\partial R - \partial S) + y\partial x(S\partial R - R\partial S)\\ + (SS - RR)\partial x\partial y + RS(S-R)y\partial x^2\end{aligned}\right\} = 0,$$

quae ut cum forma proposita conveniat, fieri oporteet

$$P = (R + S) + \frac{\partial R - \partial S}{(S-R)\partial x} \text{ et } Q = RS + \frac{S\partial R - R\partial S}{(S-R)\partial x},$$

quos valores si functiones P et Q habuerint, aequatio

$$\partial\partial y + P\partial x\partial y + Qy\partial x^2 = 0$$

per $\frac{(S-R)y\partial x}{(\partial y + Ry\partial x)(\partial y + Sy\partial x)}$ multiplicata, integrale dabit

$$\int (S-R)\partial x + l\frac{\partial y + Ry\partial x}{\partial y + Sy\partial x} = \text{Const.}$$

Si ponamus $S = M + N$ et $R = M - N$, erit

$$P = 2M - \frac{\partial N}{N\partial x} \text{ et } Q = MM - NN + \frac{\partial M}{\partial x} - \frac{M\partial N}{N\partial x}.$$

Corollarium 1.

895. Quaecunque ergo functiones ipsius x loco M et N assumantur, indeque definiantur

$$P = 2M - \frac{\partial N}{N\partial x} \text{ et } Q = MM - NN + \frac{\partial M}{\partial x} - \frac{M\partial N}{N\partial x},$$

hujus aequationis

$$\partial\partial y + P\partial x\partial y + Qy\partial x^2 = 0$$

integrale erit

$$2\int N\partial x + l\frac{\partial y + (M-N)y\partial x}{\partial y + (M+N)y\partial x} = \text{Const.}$$

Corollarium 2.

896. Si ponatur $y = e^{\int z\partial x}$, fiet nostra aequatio differentialis primi gradus

$$\partial z + zz\partial x + Pz\partial x + Q\partial x = 0,$$

cujus propterea integrale erit

$$2\int N\partial x + l\frac{z + (M-N)}{z + (M+N)} = \text{Const.}$$

Corollarium 3.

897. Si velimus, ut sit $P = 0$ et aequatio habeatur hujus-

modi $\partial\partial y + Qy\partial x^2 = 0$, capi debet $2M = \frac{\partial N}{N\partial x}$ eritque $Q = \frac{\partial M}{\partial x} - MM - NN$, ejusque aequatio integralis

$$2\int N\partial x + l\frac{\partial y + (M - N)y\partial x}{\partial y + (M + N)y\partial x} = \text{Const.}$$

Corollarium 4.

898. In genere autem prout constans capiatur vel $+\infty$ vel $-\infty$, obtinebitur integrale particulare vel

$$\partial y + (M + N)y\partial x = 0 \quad \text{vel}$$
$$\partial y + (M - N)y\partial x = 0,$$

unde erit vel

$$y = \alpha . e^{-\int(M+N)\partial x} \text{ vel } y = \beta e^{-\int(M-N)\partial x},$$

ex quibus nostrae aequationis colligitur integrale completum

$$y = e^{-\int M\partial x}\alpha\left(e^{-\int N\partial x} + \beta e^{\int N\partial x}\right).$$

Exemplum 1.

Sit $M = \alpha$ et $N = \beta$, erit $P = 2\alpha$ et $Q = \alpha\alpha - \beta\beta$, unde hujus aequationis

$$\partial\partial y + 2\alpha\partial x\partial y + (\alpha\alpha - \beta\beta)y\partial x^2 = 0$$

integrale est

$$2\beta x + l\frac{\partial y + (\alpha - \beta)y\partial x}{\partial y + (\alpha + \beta)y\partial x} = \text{Const.}$$

In quantitatibus autem finitis integrale completum

$$y = e^{-\alpha x}(A e^{-\beta x} + B e^{\beta x}).$$

Casu autem quo $\beta\beta = -\gamma\gamma$, aequatio

$$\partial\partial y + 2\alpha\partial x\partial y + (\alpha\alpha + \gamma\gamma)y\partial x^2 = 0$$

bis integrata dat

$$y = A e^{-\alpha x}\sin.(\gamma x + C).$$

At si $\gamma = 0$, aequationis

$$\partial\partial y + 2\alpha\partial x\partial y + \alpha\alpha y\partial x^2 = 0$$

integrale est

$$y = e^{-\alpha x}(A + Bx).$$

Exemplum 2.

900. Si $M = \frac{\alpha}{x}$ et $N = \beta x^n$, erit $P = \frac{2\alpha - n}{x}$, et

$$Q = \frac{\alpha\alpha}{xx} - \beta\beta x^{2n} - \frac{\alpha}{xx} - \frac{\alpha n}{xx} = \frac{\alpha(\alpha - n - 1)}{xx} - \beta\beta x^{2n}.$$

Ergo hujus aequationis

$$\partial\partial y + \frac{(2\alpha - n)\partial x \partial y}{x} + \frac{\alpha(\alpha - n - 1)y\partial x^2}{xx} - \beta\beta x^{2n} y \partial x^2 = 0$$

integrale primum est

$$\frac{2\beta}{n+1} x^{n+1} + l.\frac{x\partial y + (\alpha - \beta x^{n+1})y\partial x}{x\partial y + (\alpha + \beta x^{n+1})y\partial x} = \text{Const.}$$

Integrale autem secundum

$$y = x^{-\alpha}\left(A e^{\frac{-\beta x^{n+1}}{n+1}} + B e^{\frac{\beta x^{n+1}}{n+1}}\right):$$

si $\beta = 0$, erit id

$$y = x^{-\alpha}(A + Bx^{n+1}),$$

sin autem $\beta\beta = -\gamma\gamma$, erit

$$y = A x^{-\alpha} \sin.\left(\frac{\gamma}{n+1} x^{n+1} + C\right).$$

Corollarium 1.

901. Sumto $n = 2\alpha$, ut habeatur haec aequatio

$$\partial\partial y - \frac{\alpha(\alpha+1)y\partial x^2}{xx} - \beta\beta x^{4\alpha} y \partial x^2 = 0,$$

erit ejus integrale completum

$$y = x^{-\alpha}\left(A e^{\frac{-\beta}{2\alpha+1} x^{2\alpha+1}} + B e^{\frac{\beta}{2\alpha+1} x^{2\alpha+1}}\right),$$

si sit $\beta = 0$, erit id

$$y = x^{-\alpha}(A + Bx^{2\alpha+1}),$$

at si $\beta\beta = -\gamma\gamma$, erit hoc integrale

$$y = A x^{-\alpha} \sin. \left(\frac{\gamma}{2\alpha+1} x^{2\alpha+1} + C\right).$$

Corollarium 2.

902. Ponamus $\alpha = -1$, ut habeamus hanc aequationem $\partial\partial y - \frac{\beta\beta y \partial x^2}{x^4} = 0$, cujus ergo integrale erit

$$y = x\left(A e^{\frac{\beta}{x}} + B e^{-\frac{\beta}{x}}\right),$$

ubi notandum, si sit $\beta\beta = -\gamma\gamma$, fore $y = A x \sin. \left(\frac{\gamma}{x} + C\right)$.

Exemplum 3.

903. Ponatur $N = \frac{A x^m}{\alpha + \beta x^n}$, ut sit

$$\frac{\partial N}{N \partial x} = \frac{m}{x} - \frac{\beta n x^{n-1}}{\alpha + \beta x^n},$$

et sumatur

$$M = \frac{m}{2x} - \frac{\beta n x^{n-1}}{2(\alpha + \beta x^n)},$$

ut fiat $P = 0$ et

$$Q = \frac{-m}{2xx} - \frac{\beta n (n-1) x^{n-2}}{2(\alpha + \beta x^n)} + \frac{\beta\beta n n x^{2n-2}}{2(\alpha + \beta x^n)^2}$$
$$- \frac{mm}{4xx} + \frac{\beta m n x^{n-2}}{2(\alpha + \beta x^n)} - \frac{\beta\beta n n x^{2n-2}}{4(\alpha + \beta x^n)^2}$$
$$- \frac{AA x^{2m}}{(\alpha + \beta x^n)^2}$$

sive

$$Q = \frac{-m(m+2)}{4xx} + \frac{n(m-n+1)\beta x^{n-2}}{2(\alpha + \beta x^n)} + \frac{\beta\beta n n x^{2n-2} - 4AA x^{2m}}{4(\alpha + \beta x^n)^2},$$

et ob $\int M \partial x = \frac{1}{2} l N$, erit integrale

$$y = \frac{1}{\sqrt{N}} (C\, e^{-\int N \partial x} + D\, e^{\int N \partial x})$$

hujus aequationis $\partial\partial y + Q y \partial x^2 = 0$. Ut expressio ipsius Q fiat simplicior, hoc fieri potest pluribus modis, dum numerator partis postremae per $\alpha + \beta x^n$ divisibilis redditur.

I. Sit $m = n - 1$ et $AA = \frac{1}{4}\beta\beta nn$, eritque $Q = -\frac{(nn-1)}{4xx}$, tum vero

$$N = \frac{\frac{1}{2}\beta n x^{n-1}}{\alpha + \beta x^n} \text{ et } \int N \partial x = \tfrac{1}{2} l (\alpha + \beta x^n);$$

unde aequationis $\partial\partial y - \frac{(nn-1) y \partial x^2}{4xx} = 0$ integrale est

$$y = \frac{1}{\sqrt{x^{n-1}}} (C + D\alpha + D\beta x^n).$$

II. Sit $2m = -2$ seu $m = -1$ et $4AA = \alpha\alpha nn$, erit

$$Q = \frac{1}{4xx} - \frac{nn\beta x^{n-2}}{2(\alpha + \beta x^n)} + \frac{\beta nn x^{n-2} - \alpha nn x^{-2}}{4(\alpha + \beta x^n)}, \text{ seu}$$

$Q = \frac{1-nn}{4xx}$, ut ante.

III. Sit $2m = -n - 2$ seu

$$m = \frac{-n-2}{2} \text{ et } 4AA = -\frac{\alpha^3 nn}{\beta},$$

fietque

$$Q = \frac{-(nn-4)}{16xx} - \frac{3nn\beta x^{n-2}}{4(\alpha + \beta x^n)} + \frac{nn(\beta\beta x^{n-2} - \alpha\beta x^{-2} + \alpha\alpha x^{-n-2})}{4\beta(\alpha + \beta x^n)}$$

seu

$$Q = \frac{4 - nn}{16xx} + \frac{nn(\alpha\alpha x^{-n-2} - \alpha\beta x^{-2} - 2\beta\beta x^{n-2})}{4\beta(\alpha + \beta x^n)},$$

quae expressio abit in

$$Q = \frac{4-nn}{16xx} + \frac{nn}{4\beta}(\alpha x^{-n-2} - 2\beta x^{-2}) = \frac{4-9nn}{16xx} + \frac{nn\alpha}{4\beta x^{n+2}}.$$

Quare cum sit

$$N = +\frac{n\alpha\sqrt{-\alpha}}{2\sqrt{\beta}} \cdot \frac{x^{-\frac{n-2}{2}}}{\alpha+\beta x^n}, \text{ erit}$$

$$\int N\partial x = \frac{n\alpha\sqrt{-\alpha}}{2\sqrt{\beta}} \int \frac{\partial x}{(\alpha+\beta x^n)x^{\frac{n+2}{2}}},$$

sumatur $n = \frac{2}{3}$, ut fiat $m = -\frac{4}{3}$, $Q = \frac{\alpha}{9\beta x^{\frac{2}{3}}}$, et

$$N = \frac{\alpha\sqrt{-\alpha}}{3\sqrt{\beta}} \cdot \frac{x^{-\frac{4}{3}}}{\alpha+\beta x^{\frac{2}{3}}}, \text{ hinc}$$

$$\int N\partial x = \frac{\alpha\sqrt{-\alpha}}{3\sqrt{\beta}} \int \frac{\partial x}{(\alpha+\beta x^{\frac{2}{3}})\, x^{\frac{4}{3}}};$$

sicque aequationis

$$\partial\partial y + \frac{\alpha}{9\beta x^{\frac{2}{3}}} y\,\partial x^2 = 0$$

integrale erit

$$y = \frac{1}{\sqrt{N}}\left(Ce^{-\int N\partial x} + De^{\int N\partial x}\right).$$

Sin autem capiatur $n = -\frac{2}{3}$, ut fiat

$$m = -\frac{2}{3} \text{ et } Q = \frac{\alpha}{9\beta x^{\frac{4}{3}}}, \text{ erit}$$

$$N = \frac{-\alpha\sqrt{-\alpha}}{3\sqrt{\beta}} \cdot \frac{x^{-\frac{2}{3}}}{\alpha + \beta x^{-\frac{2}{3}}} \text{ et}$$

$$\int N \partial x = \frac{-\alpha\sqrt{-\alpha}}{3\sqrt{\beta}} \int \frac{x^{\frac{1}{3}}\partial x}{\alpha x + \beta x^{\frac{1}{3}}} = \frac{-\alpha\sqrt{-\alpha}}{3\sqrt{\beta}} \int \frac{\partial x}{\alpha x^{\frac{2}{3}} + \beta},$$

unde aequatio

$$\partial\partial y + \frac{\alpha y \partial x^2}{9\beta x^{\frac{4}{3}}} = 0$$

simili modo integratur.

Scholion 1.

904. Aequationem ergo $\partial\partial y + A x^m y \partial x^2 = 0$ his casibus integrare licuit, $m = 0$, $m = -4$, $m = -\frac{4}{3}$, $m = -\frac{8}{3}$ et $m = -2$, seu $m = -2 + \frac{2}{1}$, et $m = -2 \pm \frac{2}{3}$. Quodsi ulterius ponamus $N = \frac{A x^{\lambda}}{\alpha + \beta x^n + \gamma x^{2n}}$, simili modo integrationem casuum istius aequationis $m = -2 \pm \frac{2}{5}$ impetrabimus, quibus quoque aequatio differentialis primi gradus

$$\partial z + z z \partial x + A x^m \partial x = 0$$

integrationem admittit. Haec autem casuum integrabilium investigatio nimis est operosa, quam ut eam fusius prosequamur, praesertim cum infra methodus occurrat haec omnia commodius evolvendi.

Scholion 2.

905. Ex his colligere licet, quantus fructus ex inventione multiplicatorum, quibus etiam aequationes differentio-differentiales integrabiles redduntur, expectari queat, etiamsi exempla hic tractata tantum leve hujus methodi specimen referant. Aliquas autem sal-

$$\partial V = \frac{Ay(2P\partial y - y\partial P)}{(Byy+C+2Dx+Exx)^2} - 2y\partial y \cdot \frac{\partial Q}{\partial x},$$

cujus formulae, siquidem integrationem admittat, integrale ex variabilitate ipsius y erit

$$\frac{-AP}{B(Byy+C+2Dx+Exx)} - yy \cdot \frac{\partial Q}{\partial x} = V.$$

Sumto ergo y constante, necesse est sit

$$-\frac{A\partial P(Byy+C+2Dx+Exx)+2AP\partial x(D+Ex)}{B(Byy+C+2Dx+Exx)^2} - yy \cdot \frac{\partial\partial Q}{\partial x} = \frac{-Ayy\partial P}{(Byy+C+2Dx+Exx)^2},$$

cui satisfit, si $\frac{\partial\partial Q}{\partial x^2} = 0$ et

$$-\partial P(C+2Dx+Exx) + 2P\partial x(D+Ex) = 0,$$

quae duae conditiones an simul consistere possint, videndum est. Posterior autem dat

$$\frac{\partial P}{P} = \frac{2D\partial x + 2Ex\partial x}{C+2Dx+Exx}, \text{ ideoque } P = C+2Dx+Exx,$$

unde fit

$$Q = -\frac{\partial P}{2\partial x} = -D - Ex, \text{ hinc } \frac{\partial Q}{\partial x} = -E \text{ et } \frac{\partial\partial Q}{\partial x^2} = 0.$$

Multiplicator ergo quaesitus est

$$2\partial y(C+2Dx+Exx) - 2y\partial x(D+Ex),$$

hincque obtinetur integrale

$$\frac{\partial y^2}{\partial x^2}(C+2Dx+Exx) - \frac{2y\partial y}{\partial x}(D+Ex) - \frac{A(C+2Dx+Exx)}{B(Byy+C+2Dx+Exx)} + Eyy = \text{Const.}$$

seu utrinque addendo $\frac{A}{B}$,

$$\frac{\partial y^2}{\partial x^2}(C+2Dx+Exx) - \frac{2y\partial y}{\partial x}(D+Ex) + \frac{Ayy}{Byy+C+2Dx+Exx} + Eyy = \text{Const.}$$

Corollarium 1.

907. Haec ergo aequatio $\partial\partial y + \frac{aay\partial x^2}{(yy+xx)^2} = 0$, ubi $A = aa$, $B = 1$, $C = 0$, $D = 0$, et $E = 1$, integrabilis redditur multipli-

catore $2xx\partial y - 2yx\partial x$, ejusque integrale erit

$$\frac{xx\partial y^2}{\partial x^2} - \frac{2xy\partial y}{\partial x} + yy + \frac{aayy}{yy+xx} = bb.$$

Corollarium 2.

908. Si hic ponatur $y = ux$, ob $\partial y = u\partial x + x\partial u$, habebimus

$$xxuu + \frac{2ux^3\partial u}{\partial x} + \frac{x^4\partial u^2}{\partial x^2} + \frac{aauu}{1+uu} = bb$$
$$- 2xxuu - \frac{2ux^3\partial u}{\partial x}$$
$$+ xxuu, \text{ sive}$$

$$\frac{x^4\partial u^2}{\partial x^2} = \frac{bb+(bb-aa)uu}{1+uu}, \text{ ergo } \frac{\partial x}{xx} = \frac{\partial u\sqrt{(1+uu)}}{\sqrt{(bb+(bb-aa)uu)}},$$

unde tam x quam y per u determinatur.

Corollarium 4.

909. Simili etiam modo integratio in genere perfici potest. Sit enim brevitatis gratia

$$C + 2Dx + Exx = Bzz, \text{ erit } D + Ex = \frac{Bz\partial z}{\partial x},$$

et aequatio nostra fiet

$$\frac{Bzz\partial y^2}{\partial x^2} - \frac{2Byz\partial y\partial z}{\partial x^2} + Eyy + \frac{Ayy}{B(yy+zz)} = \frac{K}{B},$$

quae posito $y = uz$, abit in

$$\frac{Bz^4\partial u^2}{\partial x^2} - \frac{Buuzz\partial z^2}{\partial x^2} + Euuzz + \frac{Auu}{B(1+uu)} = \frac{K}{B}.$$

At $\frac{zz\partial z^2}{\partial x^2} = \frac{(D+Ex)^2}{BB}$, unde oritur

$$\frac{Bz^4\partial u^2}{\partial x^2} + \frac{CE-DD}{B}uu = \frac{K+(K-A)uu}{B(1+uu)}, \text{ seu}$$

$$\frac{BBz^4\partial u^2}{\partial x^2} = \frac{K+(K-A+DD-CE)uu+(DD-CE)u^4}{1+uu},$$

ita ut sit, restituto valore ipsius z,

$$\frac{\partial x}{C+2Dx+Exx} = \frac{\partial u\sqrt{(1+uu)}}{\sqrt{(K+(K-A+DD-CE)uu+(DD-CE)u^4)}}.$$

sicque x definitur per u, indeque etiam

$$y = uz = u\sqrt{\frac{C + 2Dx + Exx}{B}}.$$

Scholion.

910. Hinc patet substitutio, qua tam ipsa aequatio differentio-differentialis proposita, quam multiplicator ad formam commodiorem reduci debet. Posito enim ad abbreviandum $C + 2Dx + Exx = Bzz$, aequatio nostra

$$\partial\partial y - \frac{Ay\partial x^2}{B^2(yy + zz)^2} = 0,$$

ope substitutionis $y = uz$ transformatur

$$z\partial\partial u + 2\partial z\partial u + u\partial\partial z + \frac{Au\partial x^2}{BBz^3(1+uu)^2} = 0,$$

cujus multiplicator est

$$2B(zz\partial y - yz\partial z), \text{ seu } 2Bz^3\partial u,$$

vel simpliciter $z^3\partial u$. Cum autem sit $\partial z = \frac{\partial x(D+Ex)}{Bz}$, erit $\partial\partial z = \frac{E\partial x^2}{Bz} - \frac{\partial x\partial z(D+Ex)}{Bzz} = \frac{E\partial x^2}{Bz} - \frac{\partial x^2(D+Ex)^2}{BBz^3} = \frac{(CE-DD)\partial x^2}{BBz^3}$ ita ut sit $z^3\partial\partial z = \frac{CE-DD}{BB}\partial x^2$, unde aequatio nostra per $z^3\partial u$ multiplicata induit hanc formam

$$z^4\partial u\partial\partial u + 2z^3\partial z\partial u^2 + \frac{CE-DD}{BB}u\partial u\partial x^2 + \frac{Au\partial u\partial x^2}{BB(1+uu)^2} = 0,$$

manifesto integrabilem, integrali existente

$$\tfrac{1}{2}z^4\partial u^2 + \frac{CE-DD}{2BB}uu\partial x^2 - \frac{A\partial x^2}{2BB(1+uu)} = \tfrac{1}{2}\text{ Const. }\partial x^2,$$

cujus adeo nova integratio ob z functionem ipsius x mox in oculos incurrit, cum sit

$$zz\partial u = \partial x\sqrt{\left(\text{Const.} + \frac{DD-CE}{BB}uu + \frac{A}{BB(1+uu)}\right)},$$

ubi variabiles u et x sponte separantur. Caeterum hic notetur, functionem pro z assumtam satisfacere aequationi $z^3\partial\partial z = \alpha\partial x^2$, cum tamen ejus ratio non sit manifesta. Multiplicando autem hanc aequationem per $\frac{2\partial z}{z^3}$ prodit, $2\partial z\partial\partial z = \frac{2\alpha\partial x^2\partial z}{z^3}$ cujus integra-

le est $\partial z^2 = \beta \partial x^2 - \frac{\alpha \partial x^2}{zz}$, seu $\partial x = \frac{z \partial z}{\sqrt{(\beta zz - \alpha)}}$ unde porro fit

$$\beta zz = \alpha + \gamma\gamma + 2\beta\gamma x + \beta\beta xx,$$

quae est ipsa nostra forma.

Problema 113.

911. Sumto elemento ∂x constante, invenire formam generaliorem aequationum differentio-differentialium quae ope hujusmodi multiplicatoris $My\partial x + N\partial y$ integrabiles reddantur.

Solutio.

Quia multiplicator ope substitutionis $y = Ru$ in formam simplicissimam $S\partial u$ transmutari potest, hac substitutione ipsa aequatio differentio-differentialis induat hanc formam

$$\partial\partial u + P\partial x \partial u + \frac{U\partial x^2}{S} = 0,$$

cujus postremum membrum per $S\partial u$ multiplicatum sponte est integrabile, si quidem U denotet functionem quamcunque ipsius u, dum R, S et P sint functiones ipsius x. Cum ergo aequatio

$$S\partial u \partial\partial u + PS\partial x \partial u^2 + U\partial x^2 \partial u = 0,$$

debeat esse integrabilis, posito integrali

$$\tfrac{1}{2} S\partial u^2 + \partial x^2 \int U\partial u = \tfrac{1}{2} C\partial x^2,$$

necesse est sit

$$\tfrac{1}{2}\partial S \partial u^2 = PS\partial x \partial u^2, \text{ seu } P\partial x = \frac{\partial S}{2S}.$$

Quocirca haec forma generalis

$$\partial\partial u + \frac{\partial S \partial u}{2S} + \frac{U\partial x^2}{S} = 0,$$

per $S\partial u$ multiplicata dabit integrale

$$S\partial u^2 = \partial x^2 (C - 2\int U\partial u),$$

quod denuo integratum praebet

$$\int \frac{\partial x}{\sqrt{S}} = \int \frac{\partial u}{\sqrt{(C - 2\int C \partial u)}}.$$

Cum igitur haec sint manifesta, ponendo $u = \frac{y}{R}$ ad formas magis intricatas regrediamur, ite ut jam sit U = funct. $\frac{y}{R}$. Nunc vero est

$$\partial u = \frac{\partial y}{R} - \frac{y \partial R}{RR} \text{ et } \partial\partial u = \frac{\partial\partial y}{R} - \frac{2\partial R \partial y}{RR} - \frac{y\partial\partial R}{RR} + \frac{2y\partial R^2}{R^3},$$

unde aequatio nostra fit

$$\frac{\partial\partial y}{R} - \frac{2\partial R \partial y}{RR} - \frac{y\partial\partial R}{RR} + \frac{2y\partial R^2}{R^3} + \frac{U\partial x^2}{S} = 0,$$
$$+ \frac{\partial S \partial y}{2RS} \qquad - \frac{y\partial R \partial S}{2RRS},$$

quae per $\frac{S}{RR}(R\partial y - y\partial R)$ multiplicata integrabilis redditur. Ut igitur ad formam supra propositam accedamus, statuamus $S = \alpha R^4$, et aequatio

$$\frac{\partial\partial y}{R} - \frac{y\partial\partial R}{RR} + \frac{U\partial x^2}{\alpha R^4} = 0$$

per $\alpha RR(R\partial y - y\partial R)$ multiplicata integrabilis redditur. Seu haec aequatio

$$R\partial\partial y - y\partial\partial R + \frac{\partial x^2}{RR} f : \frac{y}{R} = 0$$

per $R\partial y - y\partial R$ multiplicata fit integrabilis.

Ut via ad integrationem perveniendi magis occultetur, ponatur $f : \frac{y}{R} = \frac{\alpha y}{R} + V$, ut V sit functio homogenea nullius dimensionis ipsarum y et R, ac ponatur $y\partial\partial R = \frac{\alpha y \partial x^2}{R^3}$, ut fiat

$$R\partial\partial y + \frac{V\partial x^2}{RR} = 0, \text{ seu } \partial\partial y + \frac{V\partial x^2}{R^3} = 0,$$

quae multiplicatore $R(R\partial y - y\partial R)$ redditur integrabilis. At cum sit $\partial\partial R = \frac{\alpha\partial x^2}{R^3}$, erit ut supra vidimus

$$R = \sqrt{(\alpha + 2\beta x + \gamma xx)},$$

unde dum V sit functio homogenea nullius dimensionis ipsarum y et $R = \sqrt{(\alpha + 2\beta x + \gamma xx)}$, aequatio

$$\partial\partial y + \frac{V\partial x^2}{(\alpha + 2\beta x + \gamma x x)^{\frac{3}{2}}} = 0$$

ope multiplicatoris

$$(\alpha + 2\beta x + \gamma x x)\partial y - (\beta + \gamma x) y \partial x,$$

integrabilis evadit.

Corollarium 1.

912. Posito autem $R = \sqrt{(\alpha + 2\beta x + \gamma x x)}$, aequatio nostra per $RR\partial y - Ry\partial R$ multiplicata fit

$$RR\partial y\partial\partial y - Ry\partial R\partial\partial y + \frac{V\partial x^2(R\partial y - y\partial R)}{RR} = 0,$$

cujus integrale est

$$\tfrac{1}{2}RR\partial y^2 - Ry\partial R\partial y + \int y\partial y(R\partial\partial R + \partial R^2)$$
$$+ \partial x^2 \int V\partial\frac{y}{R} = \text{Const.}\,\partial x^2,$$

ubi est

$$R\partial\partial R + \partial R^2 = \partial . R\partial R = \partial . (\beta + \gamma x)\partial x = V\partial x^2,$$

sicque integrale est

$$RR\partial y^2 - 2Ry\partial R\partial y + \gamma yy\partial x^2 \int V\partial . \frac{y}{R} = \text{Const.}\,\partial x^2.$$

Corollarium 2.

913. Quia V est functio ipsius $\frac{y}{R}$, formulae $\int V\partial . \frac{y}{R}$ integrale habetur. Pro ulteriore vero integratione posito $y = Ru$ et $\int V\partial u = U$, habebitur

$$R^4\partial u^2 - RRuu\partial R^2 + \gamma RRuu\partial x^2 + 2U\partial x^2 = G\partial x^2,$$

seu $R^4\partial u^2 = \partial x^2[G - 2U + (\beta\beta - \alpha\gamma)uu]$,

hincque

$$\frac{\partial x}{\alpha + 2\beta x + \gamma x x} = \frac{\partial u}{\sqrt{[(G - 2U + (\beta\beta - \alpha\gamma)uu]}},$$

ac porro $y = u\sqrt{(\alpha + 2\beta x + \gamma x x)}$.

Scholion.

914. Haec ergo aequatio $\partial\partial y + \frac{V\partial x^2}{R^3} = 0$, existente $R = \sqrt{(\alpha + 2\beta x + \gamma xx)}$, multo latius patet ea quam in praecedente problemate tractavimus, propterea quod hic pro V accipere licet functionem quamcunque homogeneam nullius dimensionis ipsarum y et R. Si enim sumatur $V = \frac{AR^3 y}{(myy+RR)^2}$, ipsa aequatio primum tractata oritur. Caeterum ex methodo, qua illam aequationem elicuimus apparet, eam per restrictionem ad hanc formam occultam esse perductam, cum ea aequatio, unde est nata

$$R\partial\partial y - y\partial\partial R + \frac{\partial x^2}{RR} f : \frac{y}{R} = 0$$

perspicue integrationem admittat, si per $R\partial y - y\partial R$ multiplicetur. Est enim

$$R\partial\partial y - y\partial\partial R = \partial.(R\partial y - y\partial R) \text{ et } \frac{R\partial y - y\partial R}{RR} = \partial.\frac{y}{R},$$

unde facta multiplicatione habebimus

$$(R\partial y - y\partial R)\partial.(R\partial y - y\partial R) + \partial x^2 f : \frac{y}{R}\partial.\frac{y}{R} = 0,$$

cujus aequationis utrumque membrum per se est integrabile. In aequatione autem inde eruta integrabilitas minus perspicitur, multo magis integratio est abscondita in aequationibus sequentibus.

Problema 114.

915. Sumto elemento ∂x constante, integrationem hujus aequationis

$$yy\partial\partial y + y\partial y^2 + Ax\partial x^2 = 0$$

ope multiplicatoris eam integrabilem reddentis perficere.

Solutio.

Hic frustra tentatur multiplicator hujus formae $L\partial y + M\partial x$; tentetur ergo haec forma

$$3L\partial y^2 + 2M\partial x\partial y + N\partial x^2$$

ac ponatur producti integrale

$$Lyy\partial y^3 + Myy\partial x\partial y^2 + Nyy\partial x^2\partial y + V\partial x^3 = C\partial x^3$$

cujus differentiatio perducit ad hanc aequationem

$$\begin{aligned}\partial x^3\partial V = 3Ly\partial y^4 &+ 2My\partial x\partial y^3 + Ny\partial x^2\partial y^2 + 2AMx\partial x^3\partial y + ANx\partial x^4 \\ -2Ly\partial y^4 &- yy\partial x\partial y^3\left(\tfrac{\partial L}{\partial x}\right) + 3ALx\partial x^2\partial y^2 - yy\partial x^3\partial y\left(\tfrac{\partial N}{\partial x}\right) \\ -yy\partial y^4\left(\tfrac{\partial L}{\partial y}\right) &+ 2My\partial x\partial y^3 - yy\partial x^2\partial y^2\left(\tfrac{\partial M}{\partial x}\right) \\ &- yy\partial x\partial y^3\left(\tfrac{\partial M}{\partial y}\right) - 2Ny\partial x^2\partial y^2 \\ &- yy\partial x^2\partial y^2\left(\tfrac{\partial N}{\partial y}\right)\end{aligned}$$

quae formula ut integrationem admittat, membra, quae ∂y^4, ∂y^3 et ∂y^2 continent, evanescere debent: unde primo colligitur $L - y\left(\frac{\partial L}{\partial y}\right) = 0$, ubi $\left(\frac{\partial L}{\partial y}\right)$ nascitur ex differentiatione ipsius L posito x constante. Consideretur ergo x ut quantitas constans, eritque $\frac{\partial L}{L} = \frac{\partial y}{y}$, ideoque $L = yf:x$. Negligamus autem hanc functionem ipsius x, seu ejus loco unitatem sumamus, ut sit $L = y$ et $\left(\frac{\partial L}{\partial x}\right) = 0$; secundo ergo esse debet $\left(\frac{\partial M}{\partial y}\right) = 0$. Sumamus igitur $M = 0$, etiamsi M denotare possit functionem quamcunque ipsius x, quandoquidem videbimus hoc modo negotium confici posse. Tertio itaque habebimus

$$-Ny + 3Axy - yy\left(\frac{\partial N}{\partial y}\right) = 0;$$

sumto ergo x constante, erit $3Ax\partial y = N\partial y + y\partial N$, ideoque $Ny = 3Axy$ seu $N = 3Ax$, ubi iterum functionem ipsius x, quae loco constantis ingrederetur, negligimus. Cum igitur hactenus invenimus $L = y$, $M = 0$ et $N = 3Ax$, erit $\partial V = -3Ayy\partial y + 3AAxx\partial x$, quae formula cum sponte sit integrabilis scilicet $V = -Ay^3 + AAx^3$, multiplicator nostram aequationem integrabilem reddens erit $= 3y\partial y^2 + 3Ax\partial x^2$, et producti integrale habebitur

$$y^3\partial y^3 + 3Axyy\partial x^2\partial y - Ay^3\partial x^3 + AAx^3\partial x^3 = C\partial x^3$$

quod ob constantem C est integrale completum.

Corollarium 1.

916. Hujus integralis membrum primum commode in tres factores resolvi potest. Si ponantur formulae $z^3 - A$ factores $(z - \alpha)(z - \beta)(z - \gamma)$, ut sit

$$\alpha = \sqrt[3]{A},\ \beta = \frac{-1+\sqrt{-3}}{2}\sqrt[3]{A} \text{ et } \gamma = \frac{-1-\sqrt{-3}}{2}\sqrt[3]{A},$$

erit integrale inventum

$$\left(\frac{y\partial y}{\partial x} - \alpha y + \alpha\alpha x\right)\left(\frac{y\partial y}{\partial x} - \beta y + \beta\beta x\right)\left(\frac{y\partial y}{\partial x} - \gamma y + \gamma\gamma x\right) = C,$$

existente

$$\alpha + \beta + \gamma = 0,\ \alpha\beta + \alpha\gamma + \beta\gamma = 0 \text{ et } \alpha\beta\gamma = 1.$$

Posito enim $\frac{y\partial y}{\partial x} = z$, habetur haec forma

$$z^3 + 3Axyz - Ay^3 + AAx^3,$$

cujus factor si ponatur $z - p - q$, fit

$$z^3 - 3pqz - p^3 - q^3 = 0,$$

ideoque

$$p = y\sqrt[3]{A} \text{ et } q = -x\sqrt[3]{A^2}.$$

Corollarium 2.

917. Sumto ergo constante $C = 0$, tria obtinentur integralia particularia

$$y\partial y - \alpha y\partial x + \alpha\alpha x\partial x = 0,$$

et loco α scribendo β et γ,

$$y\partial y - \beta y\partial x + \beta\beta x\partial x = 0 \text{ et}$$
$$y\partial y - \gamma y\partial x + \gamma\gamma x\partial x = 0,$$

quae posito $y = ux$ dant $\frac{\partial x}{x} = \frac{-u\partial u}{uu - \alpha u + \alpha\alpha}$, et porro integrando

$$lx = l\frac{\alpha}{\sqrt{(\alpha\alpha - \alpha u + uu)}} - \frac{1}{\sqrt{3}}\text{Ang. tang.}\frac{u\sqrt{3}}{2\alpha - u} + \text{Const.}$$

Scholion 1.

918. Aequationem autem differentialem primi ordinis inventam difficile est denuo integrare. A potestatibus quidem differentia-

lium, ponendo $\partial y = p \partial x$ et $y = ux$, unde fit $\frac{\partial x}{x} = \frac{\partial u}{p-u}$, liberari potest, prodit enim

$$x^3 (u^3 p^3 + 3 A u u p - A u^3 + A A) = C,$$

quae sumtis logarithmis, differentiata dat

$$\frac{\partial x}{x} + \frac{u u \partial p (u p p + A) + u \partial u (u p^3 + 2 A p - A u)}{u^3 p^3 + 3 A u u p - A u^3 + A A} = 0,$$

quae loco $\frac{\partial x}{x}$ scripto $\frac{\partial u}{p-u}$, abit in

$$\partial u (u p p + A)^2 + u u (p - u) \partial p (u p p + A) = 0,$$

ac per $u p p + A$ dividendo, oritur

$$A \partial u + u p p \partial u + p u u \partial p - u^3 \partial p = 0,$$

quae ponendo $p = \frac{q}{u}$ aliquanto fit simplicior, scilicet

$$A \partial u + q \partial q + q u \partial u - u u \partial q = 0,$$

cui autem posito $A = m^3$, etsi particulariter satisfacit $q = mu - mm$, tamen inde integratio completa erui vix posse videtur. Caeterum eadem haec aequatio inter p et u immediate elicitur ex aequatione differentio-differentiali proposita, quoniam in ea binae variabiles x et y ubique eundem dimensionum numerum constituunt. Posito enim $\partial y = p \partial x$ et $y = u x$, abit ea in

$$u u x \partial p + u p p \partial x + A \partial x = 0, \text{ seu } \frac{\partial x}{x} = \frac{-u u \partial p}{A + u p p} = \frac{\partial u}{p-u},$$

quae est ipsa praecedens aequatio.

Scholion 2.

919. Interim tamen aequatio proposita complete integrari potest, indeque etiam eae, quas ex ea eliciumus. Hoc autem prorsus singulari ratione praestatur, aequationem illam adeo ad differentialia tertii ordinis evehendo. Cum enim sit

$$y \partial . \frac{y \partial y}{\partial x^2} + A x \partial x = 0,$$

statuatur $\frac{\partial x}{y} = \partial v$, ut fiat

$$y \partial . \frac{\partial y}{\partial v} + A x \partial x = 0, \text{ seu } \partial . \frac{\partial y}{\partial v} + A x \partial v = 0,$$

quae sumto elemento ∂v constante, denuo differentiata praebet

$$\frac{\partial^3 y}{\partial v} + A \partial x \partial v = 0, \text{ seu } \partial^3 y + A y \partial v^3 = 0:$$

quae forma ita est comparata, ut si ei particulariter satisfaciant $y = P$, $y = Q$, $y = R$, etiam satisfaciat $y = DP + EQ + FR$. Jam vero illi satisfacit $y = e^{-\alpha v}$, si fuerit $\alpha^3 = A$, cum igitur in Coroll. 1. ternae litterae α, β, γ eadem conditione sint praeditae, habebitur integrale completum

$$y = D e^{-\alpha v} + E e^{-\beta v} + F e^{-\gamma v};$$

unde ob $A x = -\frac{\partial \partial y}{\partial v^2}$, erit

$$x = \frac{-D \alpha \alpha e^{-\alpha v} - E \beta \beta e^{-\beta v} - F \gamma \gamma e^{-\gamma v}}{A}:$$

seu mutatis constantibus, ob $A = \alpha^3 = \beta^3 = \gamma^3$,

$$x = + \mathfrak{A} e^{-\alpha v} + \mathfrak{B} e^{-\beta v} + \mathfrak{C} e^{-\gamma v}$$

$$y = - \mathfrak{A} \alpha e^{-\alpha v} - \mathfrak{B} \beta e^{-\beta v} - \mathfrak{C} \gamma e^{-\gamma v}.$$

Hinc ergo aequationis

$$A \partial u + q \partial q + q u \partial u - u u \partial q = 0$$

integrale completum his formulis continetur

$$u = \frac{-\mathfrak{A} \alpha e^{-\alpha v} - \mathfrak{B} \beta e^{-\beta v} - \mathfrak{C} \gamma e^{-\gamma v}}{\mathfrak{A} e^{-\alpha v} + \mathfrak{B} e^{-\beta v} + \mathfrak{C} e^{-\gamma v}} \text{ et}$$

$$q = \frac{\mathfrak{A} \alpha \alpha e^{-\alpha v} + \mathfrak{B} \beta \beta e^{-\beta v} + \mathfrak{C} \gamma \gamma e^{-\gamma v}}{\mathfrak{A} e^{-\alpha v} + \mathfrak{B} e^{-\beta v} + \mathfrak{C} e^{-\gamma v}},$$

ob $q = p u = \frac{y \partial y}{x \partial x} = \frac{\partial y}{x \partial v}$, quod insigne est specimen integrationis methodo directa vix perficiendae.

Problema 115.

920. Sumto elemento ∂x constante, si proponatur haec ae-

quatio $2y^3\partial\partial y + yy\partial y^2 + X\partial x^2 = 0$, existente $X = \alpha + \beta x + \gamma xx$, invenire multiplicatorem, qui eam integrabilem reddat.

Solutio.

Hic frustra tentantur multiplicatores formae

$$L\partial y + M\partial x \text{ et } L\partial y^2 + M\partial x\partial y + N\partial x^2;$$

sumamus ergo multiplicatorem hujus formae

$$2L\partial y^3 + M\partial x^2\partial y + N\partial x^3,$$

et integrale statuatur

$$Ly^3\partial y^4 + My^3\partial x^2\partial y^2 + 2Ny^3\partial x^3\partial y + S\partial x^4 = 0,$$

unde per differentiationem colligitur

$$\begin{aligned}\partial x^4\partial S = {} & 2Lyy\partial y^5 + Myy\partial x^2\partial y^3 + Nyy\partial x^3\partial y^2 + MX\partial x^4\partial y + NX\partial x^5 \\ & -3Lyy\partial y^5 + 2LX\partial x^2\partial y^3 + y^3\partial x^3\partial y^2\left(\tfrac{\partial M}{\partial x}\right) - 2y^3\partial x^4\partial y\left(\tfrac{\partial N}{\partial x}\right) \\ & -y^3\partial y^5\left(\tfrac{\partial L}{\partial y}\right) - 3Myy\partial x^2\partial y^3 - 6Nyy\partial x^3\partial y^2 \\ & -y^3\partial x^2\partial y^3\left(\tfrac{\partial M}{\partial y}\right) - 2y^3\partial x^3\partial y^2\left(\tfrac{\partial N}{\partial y}\right)\end{aligned}$$

ubi sumimus L esse functionem ipsius y tantum. Ut ergo termini ∂y^5 continentes destruantur, erit

$$-L - \frac{y\partial L}{\partial y} = 0 \text{ et } L = \frac{1}{y}.$$

Deinde pro destructione terminorum per ∂y^3 affectorum erit

$$-2Myy + \frac{2X}{y} - y^3\left(\frac{\partial M}{\partial y}\right) = 0,$$

et sumto x constante

$$\partial M + \frac{2M\partial y}{y} = \frac{2X\partial y}{y^4},$$

quae per yy multiplicata et integrata praebet

$$Myy = P - \frac{2X}{y} \text{ et } M = \frac{P}{yy} - \frac{2X}{y^3}$$

denotante P functionem quamcunque ipsius x. Jam ad terminos

∂y^2 tollendos, erit

$$-5\,N y y' - y\frac{\partial P}{\partial x} + \frac{2\,\partial X}{\partial x} - 2\,y^3\left(\frac{\partial N}{\partial y}\right) = 0,$$

et sumto x constante

$$2\,y^3\,\partial N + 5\,N\,y\,y\,\partial y = \frac{2\,\partial X}{\partial x}\,\partial y - \frac{\partial P}{\partial x}\cdot y\,\partial y,$$

quae per $\sqrt{y}$ divisa et integrata dat

$$2\,N\,y^{\frac{5}{2}} = \frac{4\,\partial X}{\partial x}\sqrt{y} - \frac{2\,\partial P}{3\,\partial x}\,y\sqrt{y},$$

neglecta functione ipsius x addenda, quoniam irrationalitas $\sqrt{y}$ in calculum non ingreditur. Erit ergo $N = \frac{2\,\partial X}{y\,y\,\partial x} - \frac{\partial P}{3\,y\,\partial x}$, ac propterea

$$\partial S = \partial y\left(\frac{PX}{yy} - \frac{2\,XX}{y^3} - \frac{4\,y\,\partial\partial X}{\partial x^2} + \frac{2\,yy\,\partial\partial P}{3\,\partial x^2}\right) + \frac{2\,X\,\partial X}{yy} - \frac{X\,\partial P}{3\,y},$$

unde fit integrando

$$S = \frac{XX}{yy} - \frac{PX}{y} - \frac{2\,yy\,\partial\partial X}{\partial x^2} + \frac{2\,y^3\,\partial\partial P}{9\,\partial x^2}$$
$$+\int\left(\frac{P\,\partial X}{y} + \frac{2\,X\,\partial P}{3\,y} + \frac{2\,yy\,\partial^3 X}{\partial x^2} - \frac{2\,y^3\,\partial^3 P}{9\,\partial x^2}\right),$$

quae finite exprimetur si $P = 0$, cum ob $X = \alpha + \beta x + \gamma x x$ fit $\partial^3 X = 0$. Quocirca habemus

$$L = \frac{1}{y},\quad M = -\frac{2\,X}{y^3},\quad \text{et}\quad N = \frac{2\,\partial X}{y\,y\,\partial x},$$

atque

$$S = \frac{XX}{yy} - \frac{2\,yy\,\partial\partial X}{\partial x^2} + \text{Const.}$$

unde aequatio integralis est

$$y^2\partial y^4 - 2X\partial x^2\partial y^2 + 4y\partial X\partial x^2\partial y + \frac{XX\,\partial x^4}{yy} - 2\,yy\,\partial x^2\,\partial\partial X = C\,\partial x^4.$$

Aequatio ergo proposita

$$2\,y^3\partial\partial y + yy\partial y^2 + \partial x^2(\alpha + \beta x + \gamma x x) = 0,$$

integrabilis redditur multiplicata per

$$\frac{2\,\partial y^3}{y} - \frac{2\,(\alpha + \beta x + \gamma x x)\,\partial x^2\,\partial y}{y^3} + \frac{2\,\partial x^3\,(\beta + 2\,\gamma x)}{yy}$$

tum vero est integrale

$$y^2\partial y^4 - 2\partial x^2\partial y^2(\alpha+\beta x+\gamma xx) + 4y\partial x^3\partial y(\beta+2\gamma x)$$
$$-4\gamma yy\partial x^4 + \frac{(\alpha+\beta x+\gamma xx)^2\partial x^4}{yy} = C\partial x^4,$$

seu

$$[yy\partial y^2 - (\alpha+\beta x+\gamma xx)\partial x^2]^2 + 4y^3\partial x^3\partial y(\beta+2\gamma x)$$
$$-4\gamma y^4\partial x^4 = Cyy\partial x^4.$$

Scholion 1.

921. Integrale hoc ita est intricatum, ut alia methodo vix inveniri potuisse videatur, verum etiam ita est comparatum, ut nulla pateat methodus id porro integrandi, unde prima integratio parum lucri attulisse est judicanda. Quemadmodum autem in praecedente problemate integrale completum ex alio fonte hausimus, ita hic simili modo integrale eruere licet, quod eo magis est notatu dignum, cum aequatio proposita in se spectata solutu sit difficillima. Ponamus scilicet itidem $\partial x = y\partial v$, et cum sit

$$\partial\partial y = \partial x\partial . \frac{\partial y}{\partial x} = y\partial v\partial . \frac{\partial y}{y\partial v},$$

erit sumendo jam elementum ∂v constans

$$\partial\partial y = y\partial v\left(\frac{\partial\partial y}{y\partial v} - \frac{\partial y^2}{yy\partial v}\right) = \partial\partial y - \frac{\partial y^2}{y}.$$

Hinc nostra aequatio induit hanc formam

$$2y^3\partial\partial y - yy\partial y^2 + yy\partial v^2(\alpha+\beta x+\gamma xx) = 0, \text{ seu}$$
$$2y\partial\partial y - \partial y^2 + \partial v^2(\alpha+\beta x+\gamma xx) = 0,$$

quae denuo differentiata praebet

$$2y\partial^3 y + y\partial v^3(\beta+2\gamma x) = 0, \text{ seu}$$
$$2\partial^3 y + \partial v^3(\beta+2\gamma x) = 0,$$

differentietur iterum, prodibitque

$$2\partial^4 y + 2\gamma y\partial v^4 = 0, \text{ seu } \partial^4 y + \gamma y\partial v^4 = 0,$$

quam aequationem si aliunde resolvere, valoremque ipsius y per v exprimere liceat, erit $x = \int y\partial v$, seu sine integratione $x = -\frac{\partial^3 y}{\gamma\partial v^3}$

$-\frac{\beta}{2\gamma}$. At manifestum est, isti aequationi differentiali quarti ordinis satisfacere $y = e^{\lambda v}$, si sit $\lambda^4 + \gamma = 0$. Ponamus ergo $\gamma = -n^4$, et quatuor ipsius λ habebuntur valores $\lambda = \pm n$ et $\lambda = \pm n \sqrt{-1}$, unde ejus integrale completum est

$$y = A e^{nv} + B e^{-nv} + C \sin. (nv + \zeta),$$

hincque

$$x = + \frac{A}{n} e^{nv} - \frac{B}{n} e^{-nv} - \frac{C}{n} \cos. (nv + \zeta) + \frac{\beta}{2n^4},$$

qui ergo valores quoque satisfacient aequationi inter x et y propositae, dummodo constantes A, B, C, et ζ ita a se pendentes capiantur, ut quantitati quoque α conveniant. His nempe valoribus substitutis fieri debet

$$\alpha + \beta x - n^4 xx + \frac{2y\partial\partial y - \partial y^2}{\partial v^2} = 0,$$

ubi tantum terminos constantes considerasse sufficit, quibus accenseri debent ii, qui quadratum sinus cosinusve anguli $nv + \zeta$ continent, quippe ex quorum combinatione quantitas constans exsurgit. Cum ergo sit

$$2y = 2 A e^{nv} + 2 B e^{-nv} + 2 C \sin. (nv + \zeta),$$
$$\frac{\partial\partial y}{\partial v^2} = nn A e^{nv} + nn B e^{-nv} - nn C \sin. (nv + \zeta),$$
$$\frac{\partial y}{\partial v} = n A e^{nv} - n B e^{-nv} + n C \cos. (nv + \zeta),$$
$$x = \frac{A}{n} e^{nv} - \frac{B}{n} e^{-nv} - \frac{C}{n} \sin. (nv + \zeta) + \frac{\beta}{2n^4},$$

erit sumtis terminis memoratis

$$\beta x = \frac{\beta\beta}{2n^4}$$
$$- n^4 xx = 2 nn AB - nn CC \cos. (nv + \zeta)^2 - \frac{\beta\beta}{4n^4},$$
$$\frac{2y\partial\partial y}{\partial v^2} = 4 nn AB - 2 nn CC \sin. (nv + \zeta)^2,$$
$$- \frac{\partial y^2}{\partial v^2} = 2 nn AB - nn CC \cos. (nv + \zeta)^2,$$

ergo

$$\alpha + 8 nn AB - 2 nn CC + \frac{\beta\beta}{4n^4} = 0, \text{ ideoque}$$
$$C = \sqrt{\left(\frac{\alpha}{2nn} + \frac{\beta\beta}{8n^6} + 4\alpha B\right)}, \text{ vel}$$

$$\alpha = 2nn(CC - 4AB) - \frac{\beta\beta}{4n^4} \text{ et}$$

$$\alpha + \beta x + \gamma xx = 2nn(CC - AB) - \left(\frac{\beta}{2nn} - nnx\right)^2.$$

Manent ergo tres constantes A, B, et ζ indeterminatae, ita ut nullum sit dubium, quin formulae pro x et y datae integrale completum exhibeant.

Scholion 2.

922. Aequationes differentio - differentiales, quas in his duobus problematibus tractavimus, ad similem formam reduci possunt. Prior enim

$$y(y\partial\partial y + \partial y^2) + X\partial x^2 = 0,$$

existente $X = Ax$ vel $X = \alpha + \beta x$, si ponatur $y\partial y = \frac{1}{2}\partial z$ seu $yy = z$, induit hanc formam

$$\tfrac{1}{2}\partial\partial z\sqrt{z} + X\partial x^2 = 0,$$

quae ope multiplicatoris $\frac{3\partial z^2}{4\sqrt{z}} + 3X\partial x^2$ integrabilis redditur. Altera vero aequatio

$$yy(2y\partial\partial y + \partial y^2) + X\partial x^2 = 0,$$

existente $X = \alpha + \beta x + \gamma xx$, posito $y = z^{\frac{2}{3}}$, fit

$$\partial y = \tfrac{2}{3}z^{-\frac{1}{3}}\partial z \text{ et } \partial\partial y = \tfrac{2}{3}z^{-\frac{1}{3}}\partial\partial z - \tfrac{2}{9}z^{-\frac{4}{3}}\partial z^2,$$

hinc $2y\partial\partial y + \partial y^2 = \frac{4}{3}z^{\frac{1}{3}}\partial\partial z$, sicque aequatio hanc induit formam $\frac{4}{3}z^{\frac{5}{3}}\partial\partial z + X\partial x^2 = 0$, quae integrabilis redditur ope hujus multiplicatoris

$$\frac{16\partial z^3}{27z^{\frac{5}{3}}} - \frac{4X\partial x^2\partial z}{3z^{\frac{2}{3}}} + \frac{2\partial X\partial x^2}{z^{\frac{4}{3}}}.$$

Hinc colligimus, pro aequatione $\partial\partial z + \frac{2X\partial x^2}{\sqrt{z}} = 0$ fore multiplicatorem $\partial z^2 + X\partial x^2 \sqrt{z}$, pro aequatione autem

$$\partial\partial z + \frac{3X\partial x^2}{4z\sqrt[3]{zz}} = 0$$

multiplicatorem fore

$$\partial z^3 - \frac{9X\partial x^2 \partial z}{4\sqrt[3]{zz}} + \tfrac{27}{8}\partial X \partial x^2 \sqrt[3]{z},$$

seu sub uno conspectu

pro aequatione	multiplicator erit
$\partial\partial z + \frac{X\partial x^2}{\sqrt{z}} = 0$	$\partial z^2 + 2X\partial x^2 \sqrt{z}$,
$\partial\partial z + \frac{X\partial x^2}{z\sqrt[3]{zz}} = 0$	$\partial z^3 + \frac{3X\partial x^4 \partial z}{\sqrt[3]{zz}} + \frac{9}{2}\partial X \partial x^2 \sqrt[3]{z}$.

Caeterum hae integrationes maxime sunt notatu dignae, cum ex aequationibus differentialibus altioribus perfici queant. Ita cum ex hac aequatione, ubi ∂v constans

$$\partial^3 y + A\partial v \partial\partial y + B\partial v^2 \partial y + Cy\partial v^3 = 0 \text{ sit}$$

$$y = \mathfrak{A}\, e^{\alpha v} + \mathfrak{B}\, e^{\beta v} + \mathfrak{C}\, e^{\gamma v},$$

si fuerint α, β, γ, radices hujus aequationis

$$r^3 + Ar^2 + Br + C = 0,$$

ponamus $\partial v = \frac{\partial x}{y}$, et cum sit

$$\partial\partial y = \partial v \partial \frac{\partial y}{\partial v} = \frac{\partial x}{y} \partial . \frac{y\partial y}{\partial x} \text{ et}$$

$$\partial^3 y = \partial v^2 \partial . \frac{\partial\partial y}{\partial v^2} = \partial v^2 \partial . \left(\frac{1}{\partial v} \partial . \frac{\partial y}{\partial v}\right) = \frac{\partial x^2}{yy} \partial . \left(\frac{y}{\partial x} \partial . \frac{y\partial y}{\partial x}\right),$$

si jam ∂x constans sumamus, erit

$$\partial\partial y = \partial\partial y + \frac{\partial y^2}{y} \text{ et}$$

$$\partial^3 y = \frac{1}{yy}\partial . y\,(y\,\partial\partial y + \partial y^2) = \partial^3 y + \frac{4\,\partial y\,\partial\partial y}{y} + \frac{\partial y^3}{yy},$$

hincque per yy multiplicando

$$yy\,\partial^3 y + 4\,y\,\partial y\,\partial\partial y + \partial y^3 + A\,\partial x\,(y\,\partial\partial y + \partial y^2)$$
$$+ B\,\partial x^2\,\partial y + C\,\partial x^3 = 0,$$

quae integrata dat

$$yy\,\partial\partial y + y\,\partial y^2 + A\,y\,\partial x\,\partial y + B\,y\,\partial x^2 + (C x + D)\,\partial x^2 = 0,$$

quae ergo per superiora integrari potest.

Problema 116.

923. Definire conditiones functionum P, Q, R et L, M, N, ut haec aequatio differentio-differentialis

$$\partial\partial y + P\,\partial y^2 + Q\,\partial x\,\partial y + R\,\partial x^2 = 0$$

integrabilis reddatur multiplicatore

$$3\,L\,\partial y^2 + 2\,M\,\partial x\,\partial y + N\,\partial x^2.$$

Solutio.

Facta multiplicatione integratio terminorum per $\partial\partial y$ affectorum dat

$$L\,\partial y^3 + M\,\partial x\,\partial y^2 + N\,\partial x^2\,\partial y,$$

quare ponatur integrale

$$L\,\partial y^3 + M\,\partial x\,\partial y^2 + N\,\partial x^2\,\partial y + V\,\partial x^3 = C\,\partial x^3,$$

cujus differentiale aequari debet formulae propositae in multiplicatorem ductae, unde oritur

$$\begin{array}{lllll}
\partial x^3\,\partial V = 3\,LP.\partial y^4 & + 3\,LQ.\partial x\,\partial y^3 & + 3\,LR.\partial x^2\,\partial y^2 & & \\
 & + 2\,MP & + 2\,MQ & + 2\,MR\,\partial x^3\,\partial y & \\
\quad -\left(\frac{\partial L}{\partial y}\right) & -\left(\frac{\partial L}{\partial x}\right) & + NP & + NQ & + NR\,\partial x^4 \\
 & -\left(\frac{\partial M}{\partial y}\right) & + \left(\frac{\partial M}{\partial x}\right) & -\left(\frac{\partial N}{\partial x}\right) & \\
 & & -\left(\frac{\partial N}{\partial y}\right). & &
\end{array}$$

Hic ergo fieri oportet

$$3\,L\,P - \left(\frac{\partial L}{\partial y}\right) = 0,$$
$$3\,L\,Q + 2\,M\,P - \left(\frac{\partial L}{\partial x}\right) - \left(\frac{\partial M}{\partial y}\right) = 0,$$
$$3\,L\,R + 2\,M\,Q + N\,P - \left(\frac{\partial M}{\partial x}\right) - \left(\frac{\partial N}{\partial y}\right) = 0.$$

Tum vero erit

$$\partial V = \left[2\,M\,R + N\,Q - \left(\frac{\partial N}{\partial x}\right)\right] \partial y + N\,R\,\partial x,$$

quae formula integrabilis esse debet. Ex illis autem aequationibus colligitur

$$P = \frac{1}{3\,L}\left(\frac{\partial L}{\partial y}\right),\ Q = \frac{1}{3\,L}\left(\frac{\partial L}{\partial x}\right) + \frac{1}{3\,L}\left(\frac{\partial M}{\partial y}\right) - \frac{2\,M}{9\,LL}\left(\frac{\partial L}{\partial y}\right) \text{ et}$$
$$R = \frac{1}{3\,L}\left(\frac{\partial M}{\partial x}\right) + \frac{1}{3\,L}\left(\frac{\partial N}{\partial y}\right) - \frac{N}{9\,LL}\left(\frac{\partial L}{\partial y}\right) - \frac{2\,M}{9\,LL}\left(\frac{\partial L}{\partial x}\right) - \frac{2\,M}{9\,LL}\left(\frac{\partial M}{\partial y}\right) + \frac{4\,MM}{27\,L^3}\left(\frac{\partial L}{\partial y}\right).$$

Corollarium 1.

924. Si L, M, et N fuerint functiones ipsius x tantum, erit $P = 0$, $Q = \frac{\partial L}{3\,L\,\partial x}$ et $R = \frac{\partial M}{3\,L\,\partial x} - \frac{2\,M\,\partial L}{9\,L\,L\,\partial x}$, hinc

$$\partial V = \left(\frac{2\,M\,\partial M}{3\,L\,\partial x} - \frac{4\,MM\,\partial L}{9\,L\,L\,\partial x} + \frac{N\,\partial L}{3\,L\,\partial x} - \frac{\partial N}{\partial x}\right)\partial y + \frac{N\,\partial M}{3\,L} - \frac{2\,M\,N\,\partial L}{9\,L\,L},$$

ac coëfficiens ipsius ∂y debet esse constans. Quare per $L^{\frac{1}{3}}$ dividendo habebitur

$$\frac{C\,\partial x}{\sqrt[3]{L}} = \frac{2\,M\,\partial M}{3\,L\sqrt[3]{L}} + \frac{N\,\partial L}{3\,L\sqrt[3]{L}} - \frac{\partial N}{\sqrt[3]{L}},$$

et integrando

$$C\int\frac{\partial x}{\sqrt[3]{L}} = \frac{M\,M}{3\,L\sqrt[3]{L}} - \frac{N}{\sqrt[3]{L}},\ \text{seu}\ N = \frac{MM}{\partial\,L} - C\,L^{\frac{1}{3}}\int\frac{\partial x}{\sqrt[3]{L}},$$

ergo

$$V = C\,y + \int\left(\frac{\partial M}{3\,L} - \frac{2\,M\,\partial L}{9\,L\,L}\right)\left(\frac{MM}{3\,L} - C\,L^{\frac{1}{3}}\int\frac{\partial x}{\sqrt[3]{L}}\right).$$

Corollarium 2.

925. Sit $M = S\sqrt[3]{L^2}$, erit

$$\partial M = \partial S \sqrt[3]{L^2} + \frac{2 S \partial L}{3\sqrt[3]{L}}, \text{ et}$$

$$V = Cy + \tfrac{1}{3}\int \frac{\partial S}{\sqrt[3]{L}} (\tfrac{1}{3} S S \sqrt[3]{L} - C L^{\frac{1}{3}} \int \frac{\partial x}{\sqrt[3]{L}}), \text{ seu}$$

$$V = Cy + \tfrac{1}{27} S^3 - \tfrac{1}{3} C \int \partial S \int \frac{\partial x}{\sqrt[3]{L}},$$

tum vero

$$N = \tfrac{1}{3} S S \sqrt[3]{L} - C L^{\frac{1}{3}} \int \frac{\partial x}{\sqrt[3]{L}} = (\tfrac{1}{3} S S - C \int \frac{\partial x}{\sqrt[3]{L}}) \sqrt[3]{L},$$

atque $P = 0$, $Q = \frac{\partial L}{3 L \partial x}$ et $R = \frac{\partial S}{3 \partial x \sqrt[3]{L}}$. Quare haec aequatio

$$\partial \partial y + \frac{\partial L \partial y}{3L} + \frac{\partial S \partial x}{3\sqrt[3]{L}} = 0$$

integrabilis redditur multiplicatore

$$3 L \partial y^2 + 2 S \partial x \partial y \sqrt[3]{L^2} + \partial x^2 (\tfrac{1}{2} S S - C \int \frac{\partial x}{\sqrt[3]{L}}) \sqrt[3]{L},$$

et integrale est

$$L \partial y^3 + S \partial x \partial y^2 \sqrt[3]{LL} + \partial x^2 \partial y (\tfrac{1}{2} S S - C \int \frac{\partial x}{\sqrt[3]{L}}) \sqrt[3]{L} + C y \partial x^3$$
$$+ \tfrac{1}{27} S^3 \partial x^3 - \tfrac{1}{3} C \partial x^3 \int \partial S \int \frac{\partial x}{\sqrt[3]{L}} = 0.$$

Corollarium 3.

926. Hic quidquid pro constante C assumatur, idem integrale prodire debet. Hinc si $C = 0$, aequationis

$$\partial \partial y + \frac{\partial L \partial y}{3 L} + \frac{\partial S \partial x}{3\sqrt[3]{L}} = 0$$

multiplicator erit

$$3\,L\,\partial y^2 + 2\,S\,\partial x\,\partial y\,\sqrt[3]{L^2} + \tfrac{1}{3}\,S\,S\,\partial x^2\,\sqrt[3]{L},$$

et integrale

$$L\,\partial y^3 + S\,\partial x\,\partial y^2\,\sqrt[3]{LL} + \tfrac{1}{3}\,SS\,\partial x^2\,\partial y\,\sqrt[3]{L} + \tfrac{1}{27}\,S^3\,\partial x^3 = D\,\partial x^3,$$

seu $(\partial y\,\sqrt[3]{L} + \tfrac{1}{3}\,S\,\partial x)^3 = D\,\partial x^3$.

Scholion 1.

927. Ex iisdem quoque conditionibus, si dentur functiones P, Q et R, definiri poterunt functiones L, M, N, quatenus quidem postrema conditio integrabilitatis patitur. Veluti si sit $P = \frac{n}{y}$, $Q = 0$ et R functio ipsius x tantum, puta $R = X$, ut habeatur haec aequatio

$$\partial\partial y + \frac{n\,\partial y^2}{y} + X\,\partial x^2 = 0,$$

cujus multiplicator si sumatur

$$3\,L\,\partial y^2 + 2\,M\,\partial x\,\partial y + N\,\partial x^2,$$

ut integrale sit

$$L\,\partial y^3 + M\,\partial x\,\partial y^2 + N\,\partial x^2\,\partial y + V\,\partial x^3 = C\,\partial x^3,$$

erit primo $\frac{3\,n\,L}{y} - \left(\frac{\partial L}{\partial y}\right) = 0$, et sumta x constante $\frac{\partial L}{L} = \frac{3\,n\,\partial y}{y}$, hinc $L = S\,y^{3\,n}$, denotante S functionem ipsius x. Deinde est

$$\frac{2\,n\,M}{y} - y^{3\,n}\,\frac{\partial S}{\partial x} - \left(\frac{\partial M}{\partial y}\right) = 0,$$

et sumta x constante

$$\partial M - \frac{2\,n\,M\,\partial y}{y} + \frac{\partial S}{\partial x}\cdot y^{3\,n}\,\partial y = 0,$$

quae per $y^{-2\,n}$ multiplicata et integrata dat

$$y^{-2\,n}\,M + \frac{\partial S}{(n+1)\,\partial x}\,y^{n+1} = T \text{ funct. ipsius } x.$$

Ergo

$$M = T\,y^{2\,n} - \frac{\partial S}{(n+1)\,\partial x}\,y^{3\,n+1}.$$

Tertio fieri debet

$$3\,SX\,y^{3n} + \frac{nN}{y} - \frac{\partial T}{\partial x}\cdot y^{2n} + \frac{\partial\partial S}{(n+1)\partial x^2}\cdot y^{3n+1} - \frac{\partial N}{\partial y} = 0,$$

unde sumta x constante

$$\partial N + \frac{nN\partial y}{y} + \frac{\partial T}{\partial x}\cdot y^{2n}\partial y - \frac{\partial\partial S}{(n+1)\partial x^2} y^{3n+1}\partial y - 3\,SX\,y^{3n}\partial y = 0,$$

quae per y^{-n} multiplicata et integrata dat

$$y^{-n}N + \frac{\partial T}{(n+1)\partial x} y^{n+1} - \frac{\partial\partial S}{2(n+1)^2\partial x^2} y^{2n+2} - \frac{3SX}{2n+1} y^{2n+1} = U f:x,$$

seu

$$N = U y^n - \frac{\partial T}{(n+1)\partial x} y^{2n+1} + \frac{\partial\partial S}{2(n+1)^2\partial x^2} y^{3n+2} + \frac{3SX}{2n+1} y^{3n+1}.$$

Ex his autem fit

$$\partial V = \partial y \left\{ \begin{array}{l} 2TXy^{2n} - \frac{2X\partial S}{(n+1)\partial x} y^{3n+1} - \frac{\partial U}{\partial x}\cdot y^n + \frac{\partial\partial T}{(n+1)\partial x^2} y^{2n+1} \\ \qquad - \frac{\partial^3 S}{2(n+1)^2\partial x^3} y^{3n+2} - \frac{3\partial . SX}{(2n+1)\partial x} y^{3n+1} \end{array} \right\}$$

$$+ X\partial x\left(U y^n - \frac{\partial T}{(n+1)\partial x} y^{2n+1} + \frac{\partial\partial S}{2(n+1)^2\partial x^2} y^{3n+2} + \frac{3SX}{2n+1} y^{3n+1}\right),$$

quae formula ut integrationem admittat, esse oportet

$$2y^{2n}\partial . TX - 2y^{3n+1}\cdot\frac{\partial . X\partial S}{(n+1)\partial x} - y^n\cdot\frac{\partial\partial U}{\partial x} + y^{2n+1}\cdot\frac{\partial^3 T}{(n+1)\partial x^2}$$
$$- y^{3n+2}\cdot\frac{\partial^4 S}{2(n+1)^2\partial x^3} - 3y^{3n+1}\frac{\partial\partial . SX}{(2n+1)\partial x} - nUXy^{n-1}\partial x$$
$$+ \frac{(2n+1)X\partial T}{u+1)} y^{2n} - \frac{(3n+2)X\partial\partial S}{2(n+1)^2\partial x} y^{3n+1} - \frac{3(3n+1)SXX\partial x}{(2n+1)} y^{3n} = 0,$$

hic ergo singulae potestates ipsius y, quatenus sunt inaequales, seorsim destrui debent. Quare potestas y^{n-1} dat $U = 0$; unde etiam potestas y^n ad nihilum redigitur. Potestas y^{2n} dat

$$(2n+2)T\partial X + (2n+2)X\partial T + (2n+1)X\partial T = 0,$$

seu $X^{2n+2}T^{4n+3} = A$; at potestas y^{2n+1} praebet $\partial^3 T = 0$, seu $T = \alpha + \beta x + \gamma xx$. Potestas vero y^{3n} postulat $S = 0$, nisi sit $n = -\frac{1}{3}$; quo casu etiam potestates y^{3n+1} et y^{3n+2} sponte evanescunt. Cum ergo sit $U = 0$, $S = 0$ et $T = \alpha + \beta x + \gamma xx$, hincque $X = B(\alpha + \beta x + \gamma xx)^{\frac{-4n-3}{2n+2}}$ haec aequatio

$$\partial\partial y + \frac{n\,\partial y^2}{y} + B(\alpha + \beta x + \gamma xx)^{\frac{-4n-3}{2n+2}}\,\partial x^2 = 0.$$

integrabilis redditur ope multiplicatoris

$$2(\alpha + \beta x + \gamma xx)\,y^{2n}\,\partial y - \frac{\partial x(\beta + 2\gamma x)}{n+1}\,y^{2n+1}.$$

Scholion 2.

928. Quanquam plurimum abest, quominus haec methodus satis adhuc sit culta, tamen specimina in hoc capite tradita abunde deolarant, quanta incrementa inde expectare queamus, unde ejus cultura maxime Geometris commendanda videtur. Quoniam igitur methodi, quibus in resolutione aequationum differentio-differentialium uti convenit, satis luculenter sunt expositae, ad sequens caput progrediamur, ubi integrationem hujusmodi aequationum, quatenus quidem id commode fieri potest, per series infinitas ostendemus.

CAPUT VII.

DE RESOLUTIONE AEQUATIONIS $\partial y y + a x^n y \partial x^2 = 0$ PER SERIES INFINITAS.

Problema 117.

929.

Sumto elemento ∂x constante, aequationem differentio-differentialem $\partial \partial y + a x^n y \partial x^2 = 0$ per seriem infinitam integrare.

Solutio.

Quaerimus hic seriem secundum potestates ipsius x progredientem, quae valorem ipsius y exprimat; et quia in altero aequationis nostrae termino quantitas x cum suo differentiali ∂x nullam, in altero vero $n + 2$ dimensiones occupat, evidens est exponentes potestatum ipsius x differentia $n + 2$ ascendere vel descendere debere

I. Ascendant primo exponentes, et fingatur series

$$y = A x^{\lambda} + B x^{\lambda + n + 2} + C x^{\lambda + 2n + 4} + \text{etc.}$$

eritque

$$\frac{\partial \partial y}{2} = \lambda(\lambda - 1) A x^{\lambda - 2} + (\lambda + n + 2)(\lambda + n + 1) B x^{\lambda + n} + \text{etc.}$$

$$a x^n y = \quad . \quad . \quad . \quad . \quad a A x^{\lambda + n}$$

unde patet primum terminum solitarium evanescere debere, ut sit $\lambda(\lambda - 1) = 0$. Quare capi oportet vel $\lambda = 0$ vel $\lambda = 1$, sic-

que duplex series obtinetur

$$y = A + B x^{n+2} + C x^{2n+4} + D x^{3n+6} + E x^{4n+8} + \text{etc.}$$
$$+ \mathfrak{A} x + \mathfrak{B} x^{n+3} + \mathfrak{C} x^{2n+5} + \mathfrak{D} x^{3n+7} + \mathfrak{E} x^{4n+9} + \text{etc.}$$

substitutione ergo facta fieri oportet

$$0 = (n+2)(n+1)Bx^{n} + (2n+4)(2n+3)Cx^{2n+2} + (3n+6)(3n+5)Dx^{3n+4} + \text{etc.}$$
$$+ aA \qquad + \qquad aB \qquad\qquad + aC$$

$$0 = (n+3)(n+2)\mathfrak{B}x^{n+1} + (2n+5)(2n+4)\mathfrak{C}x^{2n+3} + (3n+7)(3n+6)\mathfrak{D}x^{3n+5} + \text{etc.}$$
$$+ a\mathfrak{A} \qquad\qquad + a\mathfrak{B} \qquad\qquad + a\mathfrak{C}$$

unde litteris A et $\mathfrak{A}$ arbitrio nostro relictis, reliquae per eas ita determinantur

$$B = \frac{-aA}{(n+1)(n+2)};\; C = \frac{-aB}{2(2n+3)(n+2)};\; D = \frac{-aC}{3(3n+5)(n+2)} \text{ etc.}$$
$$\mathfrak{B} = \frac{-a\mathfrak{A}}{(n+3)(n+2)};\; \mathfrak{C} = \frac{-a\mathfrak{B}}{2(2n+5)(n+2)};\; \mathfrak{D} = \frac{-a\mathfrak{C}}{3(3n+7)(n+2)} \text{ etc.}$$

sicque habebitur integrale completum ita expressum

$$y = A - \frac{aAx^{n+2}}{1(n+1).(n+2)} + \frac{a^2Ax^{2n+4}}{1.2(n+1)(2n+3).(n+2)^2}$$
$$- \frac{a^2Ax^{3n+6}}{1.2.3(n+1)(2n+3)(3n+5).(n+2)^3} + \text{etc.}$$
$$+ \mathfrak{A}x - \frac{a\mathfrak{A}x^{n+3}}{1(n+3).(n+2)} + \frac{a^2\mathfrak{A}x^{2n+5}}{1.2(n+3)(2n+5).(n+2)^2}$$
$$- \frac{a^3\mathfrak{A}x^{3n+7}}{1.2.3(n+3)(2n+5)(3n+7).(n+2)^3} + \text{etc.}$$

II. Descendant jam exponentes, et ficta serie

$$y = Ax^{\lambda} + Bx^{\lambda-n-2} + Cx^{\lambda-2n-4} + \text{etc.}$$

habebitur

$$\frac{\partial\partial y}{\partial x^2} = \lambda(\lambda-1)Ax^{\lambda-2} + (\lambda-n-2)(\lambda-n-3)Bx^{\lambda-n-4} + \text{etc.}$$
$$ax^n y = aAx^{\lambda+n} + aBx^{\lambda-2} + \text{etc.}$$

ubi cum terminus $x^{\lambda+n}$ sui similem non habeat, tolli nequit, ita ut hinc nulla aequationis resolutio obtineatur.

Corollarium 1.

930. Geminata series pro y inventa, quoniam litterae A et $\mathfrak{A}$ arbitrio nostro relinquuntur, integrale completum aequationis differentio-differentialis $\partial\partial y + ax^n y \partial x^2 = 0$ exhibet; tribuendo autem litteris A et $\mathfrak{A}$ datos valores, integralia particularia nascentur.

Corollarium 2.

931. Si ponamus $n+2=m$, seu $n=m-2$, hujus aequationis $\partial\partial y + ax^{m-2} y \partial x^2 = 0$ integrale completum ita commodius exprimetur

$$y = \begin{cases} A - \dfrac{aAx^m}{1(m-1).m} + \dfrac{a^2Ax^{2m}}{1.2(m-1)(2m-1).m^2} \\ \quad - \dfrac{a^3Ax^{3m}}{1.2.3(m-1)(2m-1)(3m-1).m^3} + \text{etc.} \\ \mathfrak{A}x - \dfrac{a\mathfrak{A}x^{m+1}}{1(m+1).m} + \dfrac{a^2\mathfrak{A}x^{2m+1}}{1.2(m+1)(2m+1).m^2} \\ \quad - \dfrac{a^3\mathfrak{A}x^{3m+1}}{1.2.3(m+1)(2m+1)(3m+1).m^3} + \text{etc.} \end{cases}$$

Corollarium. 3.

932. Si exponens m fuerit positivus et unitate major, hae series eo magis convergunt, quo minor valor quantitati x tribuatur: aliis vero casibus in praxi hae series adhiberi nequeunt, nisi forte eae ipsae in alias convergentes transformari possint.

Scholion 1.

933. Dantur tamen casus, quibus hae series omni plane usu destituuntur, quod evenit, si quispiam factorum denominatores con-

stituentium evanescat, sicque omnes termini sequentes in infinitum excrescant, quibus casibus series in alias formas transmutari convenit. Hic primo occurrit casus $m = 0$ seu $n = -2$, quo utriusque seriei omnes termini praeter primos fiunt infiniti, hoc vero casu aequatio, quae est $\partial\partial y + \frac{a y \partial x^2}{x^2} = 0$, cum sit homogenea, singularem integrationem admittit: inveniri enim potest potestas ipsius x, quae pro y substituta aequationi satisfacit. Ponatur scilicet $y = x^\lambda$, prodibitque

$$\lambda(\lambda - 1) x^{\lambda - 2} + a x^{\lambda - 2} = 0, \text{ seu } \lambda\lambda - \lambda + a = 0$$

unde colligitur $\lambda = \frac{1}{2} \pm \sqrt{(\frac{1}{4} - a)}$, ob quem duplicem valorem est integrale completum

$$y = A x^{\frac{1}{2} + \sqrt{(\frac{1}{4} - a)}} + B x^{\frac{1}{2} - \sqrt{(\frac{1}{4} - a)}},$$

quae aequatio casu $a > \frac{1}{4}$ abit in hanc formam

$$y = A x^{\frac{1}{2}} \sin. [(a - \tfrac{1}{4})^{\frac{1}{2}} l x + \alpha];$$

unde patet, casu $a = \frac{1}{4}$ fore

$$y = (A + B l x) \sqrt{x}.$$

Scholion 2.

934. Reliqui casus ad incommodum ducentes sunt, si vel $m = \frac{1}{i}$ vel $m = -\frac{1}{i}$ denotante i numerum quemcunque integrum. Casu $m = \frac{1}{i}$ prior tantum series fit incongrua, casu vero $m = -\frac{1}{i}$ posterior tantum. Quare illo casu ponendo $A = 0$ hoc vero $\mathfrak{A} = 0$, series saltem una idonea habetur, integrale particulare exhibens. Verum cognito integrali particulari quod sit $y = P$, inde aequationis

$$\partial\partial y + a x^{m-2} y \partial x^2 = 0$$

integrale completum eruitur ponendo $y = P z$, unde fit

$$P \partial \partial z + 2 \partial P \partial z + z \partial \partial P + a x^{m-2} P z \partial x^2 = 0,$$

at per hypothesin est

$$\partial \partial P + a x^{m-2} P \partial x^2 = 0,$$

ergo prodit

$$P \partial \partial z + 2 \partial P \partial z = 0, \text{ seu}$$

$$P P \partial z = C \partial x \text{ et } z = C \int \frac{\partial x}{P P}.$$

Cum autem P sit series infinita, hinc valorem ipsius z cognoscere haud licet. At casibus illis memoratis pars integralis logarithmum ipsius x involvit, quod vel inde intelligitur quod $\frac{x^0}{0}$ aequivaleat ipsi $l x$. Quare in aequatione

$$\partial \partial y + a x^{m-2} y \partial x^2 = 0,$$

ponendo $y = p + q l x$, ob $\partial y = \partial p + \frac{q \partial x}{x} + \partial q l x$, erit

$$\partial \partial p + \frac{2 \partial x \partial q}{x} - \frac{q \partial x^2}{x x} + \partial \partial q l x + a p x^{m-2} \partial x^2$$
$$+ a q x^{m-2} \partial x^2 l x = 0,$$

in qua partes $l x$ involventes seorsim destruantur necesse est; ita ut hae binae habeantur aequationes

$$\partial \partial q + a q x^{m-2} \partial x^2 = 0 \text{ et}$$
$$\partial \partial p + \frac{2 \partial x \partial q}{x} - \frac{q \partial x^2}{x x} + a p x^{m-2} \partial x^2 = 0,$$

ubi pro q ea binarum superiorum serierum accipi debet, quae casu oblato incommodo caret, eaque constituta ex posteriori aequatione facile quantitas p per seriem exprimetur. Hujusmodi casus in sequentibus exemplis evolvamus; tantum notemus illam operationem perinde se habere, etiamsi loco $l x$ sumatur $l x + \alpha$, ita ut inventis p et q futurum sit

$$y = \alpha q + p + q l x, \text{ seu } y = p + q l \beta x.$$

Exemplum 1.

935. *Posito* $m = 1$, *hanc aequationem*

$$\partial\partial y + \frac{ay\partial x^2}{x} = 0$$

per series resolvere.

Posito $y = p + q\,lx$, capi oportet

$$q = \mathfrak{A}x - \frac{a\mathfrak{A}x^2}{1.2} + \frac{a^2\mathfrak{A}x^3}{1.2.2.3} - \frac{a^3\mathfrak{A}x^4}{1.2.2.3.3.4} + \text{etc.}$$

pro qua ponamus brevitatis gratia

$$q = \mathfrak{A}x + \mathfrak{B}x^2 + \mathfrak{C}x^3 + \mathfrak{D}x^4 + \text{etc.}$$

tum vero quaeratur p ex hac aequatione

$$\frac{\partial\partial p}{\partial x^2} + \frac{2\partial q}{x\partial x} - \frac{q}{xx} + \frac{ap}{x} = 0:$$

fingamus ergo

$$p = \text{A} + \text{B}x + \text{C}x^2 + \text{D}x^3 + \text{E}x^4 + \text{etc.}$$

eritque facta substitutione

$$\left.\begin{array}{llllll} & 2\text{C} & +6\text{D}x & +12\text{E}xx & +20\text{F}x^3 & +30\text{G}x^4 \\ +2\mathfrak{A}\frac{1}{x} & +4\mathfrak{B} & +6\mathfrak{C} & +8\mathfrak{D} & +10\mathfrak{E} & +12\mathfrak{F} \\ -\mathfrak{A} & -\mathfrak{B} & -\mathfrak{C} & -\mathfrak{D} & -\mathfrak{E} & -\mathfrak{F} \\ +a\text{A} & +a\text{B} & +a\text{C} & +a\text{D} & +a\text{E} & +a\text{F} \end{array}\right\} = 0$$

Cum jam dentur coëfficientes $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$ etc. erit $\text{A} = -\frac{\mathfrak{A}}{a}$; quantitas B non determinatur, tum vero

$$\text{C} = \frac{-3\mathfrak{B}}{2} - \frac{a\text{B}}{1.2} = \frac{+3a\mathfrak{A}}{1^3.2^2} - \frac{a\text{B}}{1.2},$$

$$\text{D} = \frac{-5\mathfrak{C}}{6} - \frac{a\text{C}}{2.3} = \frac{-5a^2\mathfrak{A}}{1^2.2^3.3^2} - \frac{a\text{C}}{2.3},$$

$$\text{E} = \frac{-7\mathfrak{D}}{12} - \frac{a\text{D}}{3.4} = \frac{+7a^3\mathfrak{A}}{1^2.2^2.3^3.4^2} - \frac{a\text{D}}{3.4},$$

$$\text{F} = \frac{-9\mathfrak{E}}{20} - \frac{a\text{E}}{4.5} = \frac{-9a^4\mathfrak{A}}{1^2.2^2.3^2.4^3.5^2} - \frac{a\text{E}}{4.5}, \text{ etc.}$$

ubi pro B scribere licet 0, quandoquidem in integrali $y = p + q\,lx$ addimus partem αq, quae ex littera B oritur, ita ut sit

$$p = \text{A} + \text{C}x^2 + \text{D}x^3 + \text{E}x^4 + \text{F}x^5 + \text{etc.}$$

Hinc erit

$$C = \frac{3\,a\mathfrak{A}}{1^3.2^2},\quad D = \frac{-14\,a^2\mathfrak{A}}{1^3.2^3.3^2},\quad E = \frac{+70\,a^3\mathfrak{A}}{1^3.2^3.3^3.4^2},\quad F = \frac{-404\,a^4\mathfrak{A}}{1^3.2^3.3^3.4^3.5^2},\ \text{etc.}$$

ubi notetur esse

$$14 = 3.3 + 5.1,\quad 70 = 4.14 + 7.1.2,\quad 404 = 5.70 + 9.1.2.3,$$

et pro sequente $2688 = 6.404 + 11.1.2.3.4$; estque

$$y = p + aq + qlx.$$

Exemplum 2.

936. *Posito* $m = -1$ *hanc aequationem*

$$\partial\partial y + \frac{ay\partial x^2}{x^3} = 0,$$

per series resolvere.

Posito $y = p + aq + qlx$, capi oportet

$$q = A - \frac{aA}{1.2x} + \frac{a^2A}{1.2^2.3x^2} - \frac{a^3A}{1.2^2.3^2.4x^3} + \text{etc.}$$

pro qua ponatur brevitatis gratia

$$q = A + \frac{B}{x} + \frac{C}{x^2} + \frac{D}{x^3} + \frac{E}{x^4} + \text{etc.}$$

tum vero quantitas p ex hac aequatione definiri debet

$$\partial\partial p + \frac{2\partial q\partial x}{x} - \frac{q\partial x^2}{xx} + \frac{ap\partial x^2}{x^3} = 0.$$

Fingamus ergo

$$p = \mathfrak{A}x + \mathfrak{B} + \frac{\mathfrak{C}}{x} + \frac{\mathfrak{D}}{xx} + \frac{\mathfrak{E}}{x^3} + \frac{\mathfrak{F}}{x^4} + \text{etc.}$$

unde facta substitutione prodit

$$\left.\begin{array}{cccccc}
\frac{a\mathfrak{A}}{xx} + & \frac{a\mathfrak{B}}{x^3} + & \frac{a\mathfrak{C}}{x^4} + & \frac{a\mathfrak{D}}{x^5} + & \frac{a\mathfrak{E}}{x^6} + & \frac{a\mathfrak{F}}{x^7} + \text{etc.} \\
-A - & B - & C - & D - & E - & F \\
 & -2B - & 4C - & 6D - & 8E - & 10F \\
 & +2\mathfrak{C} + & 6\mathfrak{D} + & 12\mathfrak{E} + & 20\mathfrak{F} + & 30\mathfrak{G}
\end{array}\right\} = 0;$$

et coëfficientes $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$, etc. ita determinantur, ut sit

$\mathfrak{A} = \frac{A}{a}$; secundus $\mathfrak{B}$ non definitur, tum vero est

$$\mathfrak{C} = \frac{3\,B}{1.\,2} - \frac{a\mathfrak{B}}{1.\,2} = \frac{-3\,a\,A}{1^3.\,2^2} - \frac{a\,\mathfrak{B}}{1.\,2},$$

$$\mathfrak{D} = \frac{5\,C}{2.\,3} - \frac{a\,\mathfrak{C}}{2\;3} = \frac{5\,a^2\,A}{2^3.\,3^2} - \frac{a\,\mathfrak{C}}{2.\,3},$$

$$\mathfrak{E} = \frac{7\,D}{3.\,4} - \frac{a\,\mathfrak{D}}{3.\,4} = \frac{-\,a^3\,A}{2^2.\,3^3.\,4^2} - \frac{a\,\mathfrak{D}}{3.\,4},$$

$$\mathfrak{F} = \frac{9\,E}{4.\,5} - \frac{a\,\mathfrak{E}}{4.\,5} = \frac{9\,a^4\,A}{2^2.\,3^2.\,4^3.\,5^2} - \frac{a\,\mathfrak{E}}{4.\,5}, \text{ etc.}$$

si sumatur, id quod sine detrimento generalitatis fieri licet, $\mathfrak{B} = 0$, ita ut sit

$$p = \mathfrak{A}\,x + \frac{\mathfrak{C}}{x} + \frac{\mathfrak{D}}{xx} + \frac{\mathfrak{E}}{x^3} + \frac{\mathfrak{F}}{x^4} + \text{etc. erit}$$

$$\mathfrak{C} = \frac{-3\,a\,A}{1^3.\,2^2},\; \mathfrak{D} = \frac{+\,4\,a^2\,A}{1^3.\,2^3.\,3^2},\; \mathfrak{E} = \frac{-\,70\,a^3\,A}{1^3.\,2^3.\,3^3.\,4^2},$$

$$\mathfrak{F} = \frac{+\,404\,a^4\,A}{1^3.\,2^3.\,3^3.\,4^3.\,5^2}, \text{ etc.}$$

qui valores similes sunt praecedentibus.

Exemplum 3.

937. *Posito* $m = \frac{1}{2}$, *hanc aequationem*

$$\partial\partial y + \frac{a\,y\,\partial x^2}{x\sqrt{x}} = 0$$

per series resolvere.

Posito $y = p + aq + qlx$, capi oportet

$$q = \mathfrak{A}x - \frac{4\,a\,\mathfrak{A}}{1.\,3}\,x^{\frac{3}{2}} + \frac{16\,a^2\,\mathfrak{A}}{1.\,2.\,3.\,4}\,x^2 - \frac{64\,a^3\,\mathfrak{A}}{1.\,2.\,3.\,3.\,4.\,5}\,x^{\frac{5}{2}}$$
$$+ \frac{256\,a^4\,\mathfrak{A}}{1.\,2.\,3.\,4.\,3.\,4.\,5.\,6}\,x^3 - \text{etc.}$$

pro qua brevitatis gratia scribatur

$$q = \mathfrak{A}x + \mathfrak{B}x^{\frac{3}{2}} + \mathfrak{C}x^2 + \mathfrak{D}x^{\frac{5}{2}} + \mathfrak{E}x^3 + \mathfrak{F}x^{\frac{7}{2}} + \mathfrak{G}x^4 + \text{etc.}$$

tum vero quantitas p ex hac aequatione definiri debet:

$$\partial\partial p + \frac{2\,\partial x\,\partial q}{x} - \frac{q\,\partial x^2}{x\,x} + \frac{a\,p\,\partial x^2}{x\sqrt{x}} = 0.$$

Fingamus ergo

$$p = \Delta + Ax^{\frac{1}{2}} + Bx + Cx^{\frac{3}{2}} + Dx^2 + Ex^{\frac{5}{2}} + Fx^3 + \text{etc.}$$

prodibitque facta substitutione

$$\left.\begin{array}{llllllll}
\frac{a\Delta}{x\sqrt{x}} & + \frac{aA}{x} & + \frac{aB}{\sqrt{x}} & + aC & + aD\sqrt{x} & + aEx & + aFx\sqrt{x} \\
 & - \mathfrak{A} & - \mathfrak{B} & - \mathfrak{C} & - \mathfrak{D} & - \mathfrak{E} & - \mathfrak{F} \\
 & + 2\mathfrak{A} & + 3\mathfrak{B} & + 4\mathfrak{C} & + 5\mathfrak{D} & + 6\mathfrak{E} & + 7\mathfrak{F} \\
 - \frac{A}{4} & + 0 & + \frac{3}{4}C & + 2D & + \frac{15}{4}E & + 6F & + \frac{35}{4}G
\end{array}\right\} = 0.$$

Hinc colligitur fore

$$A = \frac{-\mathfrak{A}}{a}, \quad \Delta = \frac{-\mathfrak{A}}{4aa},$$

et B non determinatur, porro

$$C = \frac{-4aB}{1.3} - \frac{8\mathfrak{B}}{1.3} = \frac{-4aB}{1.3} + \frac{32a\mathfrak{A}}{1^2.3^2},$$

$$D = \frac{-4aC}{2.4} - \frac{12\mathfrak{C}}{2.4} = \frac{-4aC}{2.4} - \frac{12.16.a^2\mathfrak{A}}{1.3.2^2.4^2},$$

$$E = \frac{-4aD}{3.5} - \frac{16\mathfrak{D}}{3.5} = \frac{-4aD}{3.5} + \frac{16.64.a^3\mathfrak{A}}{1.3.2.4.3^2.5^2},$$

$$F = \frac{-4aE}{4.6} - \frac{20\mathfrak{E}}{4.6} = \frac{-4aE}{4.6} - \frac{20.256.a^4\mathfrak{A}}{1.3.2.4.3.5.4^2.6^2}, \text{ etc.}$$

Quodsi jam ponatur $B = 0$ ut sit

$$p = \frac{-\mathfrak{A}}{4aa} - \frac{\mathfrak{A}}{a}\sqrt{x} + * + Cx\sqrt{x} + Dx^2 + Ex^2\sqrt{x} + Fx^3 + \text{etc.}$$

erit

$$C = \frac{8.4a\mathfrak{A}}{1^2.3^2}, \quad D = \frac{-100.16.a^2\mathfrak{A}}{1^2.3^2.2^2.4^2}, \quad E = \frac{1884.64.a^3\mathfrak{A}}{1^2.3^2.2^2.4^2.3^2.5^2},$$

$$F = \frac{-52416.256a^4\mathfrak{A}}{1^2.3^2.2^2.4^2.3^2.5^2.4^2.6^2}, \text{ etc.}$$

ubi notetur esse

$$100 = 2.4.8 + 1.3.12, \quad 1884 = 3.5.100 + 1.3.2.4.16,$$
$$52416 = 4.6.1884 + 1.3.2.4.3.5.20.$$

Exemplum 4.

938. *Posito* $m = -\frac{1}{2}$, *hanc aequationem*

$$\partial\partial y + \frac{ay\partial x^2}{xx\sqrt{x}} = 0$$

per series resolvere.

Posito $y = p + aq + qlx$, capi oportet

$$q = A - \frac{4aA}{1.3}x^{-\frac{1}{2}} + \frac{16a^2A}{1.3.2.4}x^{-1} - \frac{64a^3A}{1.3.2.4.3.5}x^{-\frac{3}{2}} + \text{etc.}$$

pro qua brevitatis causa scribamus

$$q = A + Bx^{-\frac{1}{2}} + Cx^{-1} + Dx^{-\frac{3}{2}} + Ex^{-2} + Fx^{-\frac{5}{2}} + \text{etc}$$

et littera p ex hac aequatione definiri debet

$$\partial\partial p + \frac{2\partial x\partial q}{x} - \frac{q\partial x^2}{xx} + \frac{ap\partial x^2}{xx\sqrt{x}} = 0.$$

Fingamus

$$p = \Delta x + \mathfrak{A}\sqrt{x} + \mathfrak{B} + \mathfrak{C}x^{-\frac{1}{2}} + \mathfrak{D}x^{-1} + \mathfrak{E}x^{-\frac{3}{2}}$$
$$+ \mathfrak{F}x^{-2} + \mathfrak{G}x^{-\frac{5}{2}} + \text{etc.}$$

et facta substitutione prodit

$$\left.\begin{array}{ccccccc}
\frac{\mathfrak{A}\Delta}{x\sqrt{x}} & + \frac{a\mathfrak{A}}{xx} & + \frac{a\mathfrak{B}}{xx\sqrt{x}} & + \frac{a\mathfrak{C}}{x^3} & + \frac{a\mathfrak{D}}{x^3\sqrt{x}} & + \frac{a\mathfrak{E}}{x^4} & + \frac{a\mathfrak{F}}{x^4\sqrt{x}} \\
 & - A & - B & - C & - D & - E & - F \\
 & & - B & - 2C & - 3D & - 4E & - 5F \\
-\frac{1}{4}\mathfrak{A} & & + \frac{3}{4}\mathfrak{C} & + 2\mathfrak{D} & + \frac{15}{4}\mathfrak{E} & + 6\mathfrak{F} & + \frac{35}{4}\mathfrak{G}
\end{array}\right\} = 0,$$

unde sequentes determinationes colliguntur

$$\mathfrak{A} = \frac{A}{a} \text{ et } \Delta = \frac{\mathfrak{A}}{4a} = \frac{A}{4aa},$$

at $\mathfrak{B}$ non determinatur: porro

$$\mathfrak{C} = \frac{-4a\mathfrak{B}}{1.3} + \frac{8B}{1.3} = \frac{-4a\mathfrak{B}}{1.3} - \frac{8.4aA}{1^2.3^2},$$

$$\mathfrak{D} = \frac{-4a\mathfrak{C}}{2.4} + \frac{12C}{2.4} = \frac{-4a\mathfrak{C}}{2.4} + \frac{12.16a^2A}{1.3.2^2.4^2},$$

$$\mathfrak{E} = \frac{-4a\mathfrak{D}}{3.5} + \frac{16D}{3.5} = \frac{-4a\mathfrak{D}}{3.5} - \frac{16.64a^3A}{1.3.2.4.3^2.5^2},$$

$$\mathfrak{F} = \frac{-4a\mathfrak{E}}{4.6} + \frac{20E}{4.6} = \frac{-4a\mathfrak{E}}{4.6} + \frac{20.256a^4A}{1.3.2.4.3.5.4^2 6^2}, \text{ etc.}$$

Quodsi jam sumatur $\mathfrak{B} = 0$, erit

$$\mathfrak{C} = \frac{-8.4aA}{1^2.3^2}, \quad \mathfrak{D} = \frac{+100.16a^2A}{1^2.3^2.2^2.4^2}, \quad \mathfrak{E} = \frac{-1884.64a^3A}{1^2.3^2.2^2.4^2.3^2.5^2}, \text{ etc.}$$

qui numeri ut ante progrediuntur.

Scholion.

939. Ex his exemplis perspicitur, quomodo series aequationem

$$\partial\partial y + a x^{m-2} y \partial x^2 = 0$$

resolventes in reliquis casibus, quibus $m = \pm \frac{1}{i}$, inveniri oporteat; ubi observetur, si sit $m = +\frac{1}{i}$, pro q hanc seriem accipi debere

$$q = \mathfrak{A}x + \mathfrak{B}x^{1+\frac{1}{i}} + \mathfrak{C}x^{1+\frac{2}{i}} + \mathfrak{D}x^{1+\frac{3}{i}} + \text{etc.}$$

tum vero formam ipsius p tali serie exprimi

$$p = A + Bx^{\frac{1}{i}} + Cx^{\frac{2}{i}} + Dx^{\frac{3}{i}} + \text{etc.}$$

cujus coëfficientes ex superioribus ut ante definiantur. Sin autem sit $m = -\frac{1}{i}$, pro q sumatur series

$$q = A + Bx^{-\frac{1}{i}} + Cx^{-\frac{2}{i}} + Dx^{-\frac{3}{i}} + \text{etc.}$$

at pro p hujusmodi formam accipi conveniet

$$p = \mathfrak{A}x + \mathfrak{B}x^{1-\frac{1}{i}} + \mathfrak{C}x^{1-\frac{2}{i}} + \mathfrak{D}x^{1-\frac{3}{i}} + \text{etc.}$$

unde pariter singulos coëfficientes uno excepto determinare licebit. Atque hoc artificium in genere est tenendum, quoties in resolutione aequationis generalis ad series pervenitur, cujus coëfficientes certis casibus in infinitum excrescunt, quod plerumque indicio est, logarithmos esse introducendos. Verum etiam eadem aequatio $\partial\partial y + a x^n y \partial x^2 = 0$ aliis modis per series resolvi potest, dum ea ante resolutionem in aliam formam transmutatur, ubi cum evenire possit, ut series certis casibus abrumpatur, quibus adeo integrale revera assignari potest, talem transformationem maxime notabilem hic explicemus.

Problema 118.

940. Aequationem differentio-differentialem

$$\partial\partial y + a x^n y \partial x^2 = 0$$

in aliam formam transfundere, cujus resolutio per series infinitas commode institui possit.

Solutio.

Utamur substitutione $y = e^{\int p \partial x} z$, ubi p sit certa functio ipsius x aequationem commode resolubilem suppeditans. Erit ergo

$$\partial y = e^{\int p \partial x}(\partial z + p z \partial x) \text{ et}$$

$$\partial\partial y = e^{\int p \partial x}(\partial\partial z + 2 p \partial x \partial z + z \partial x \partial p + p p z \partial x^2),$$

unde aequatio proposita abit in

$$\partial\partial z + 2 p \partial x \partial z + z \partial x \partial p + p p z \partial x^2 + a x^n z \partial x^2 = 0,$$

ubi p ita capiatur, ut fiat

$$p p + a x^n = 0, \text{ seu } p = x^{\frac{n}{2}} \sqrt{-a}.$$

Ponamus ideo $a = -c c$ et $n = 2m$, ut proposita sit haec aequatio

$$\partial\partial y - c c x^{2m} y \partial x^2 = 0,$$

quae posito

$$p = c x^m \text{ et } y = e^{\int p \partial x} z = e^{\frac{c}{m+1} x^{m+1}} z,$$

induet hanc formam

$$\partial\partial z + 2 c x^m \partial x \partial z + m c x^{m-1} z \partial x^2 = 0,$$

in qua, cum x occupet vel nullam, vel $m + 1$ dimensiones, fingamus ipsius z valorem

$$z = A x^\lambda + B x^{\lambda+m+1} + C x^{\lambda+2m+2} + \text{etc.}$$

quo substituto fit

$$\begin{array}{ll} \lambda(\lambda-1)Ax^{\lambda-2} & +(\lambda+m+1)(\lambda+m)Bx^{\lambda+m-1}=0, \\ & +2\lambda Ac \\ & +mAc \end{array}$$

unde perspicuum est sumi debere vel $\lambda=0$ vel $\lambda=1$. Consequimur ergo seriem duplicatam hujusmodi

$$\begin{array}{ll} z=A & +Bx^{m+1}+Cx^{2m+2}+Dx^{3m+3}+Ex^{4m+4}+\text{etc.} \\ & \mathfrak{A}x+\mathfrak{B}x^{m+2}+\mathfrak{C}x^{2m+3}+\mathfrak{D}x^{3m+4}+\mathfrak{E}x^{4m+5}+\text{etc.} \end{array}$$

qua substituta fit

$$\left.\begin{array}{lll} (m+1)mBx^{m-1} & +2(m+1)(2m+1)Cx^{2m} & +3(m+1)(3m+2)Dx^{3m+1} \\ +mAc & +2(m+1)Bc & +4(m+1)Cc \\ & +mBc & +mCc \end{array}\right\}=0$$

$$\left.\begin{array}{lll} +(m+1)(m+2)\mathfrak{B}x^{m} & +2(m+1)(2m+3)\mathfrak{C}x^{2m+1} & +3(m+1)(3m+4)\mathfrak{D}x^{3m+2} \\ +2\mathfrak{A}c & +2(m+2)\mathfrak{B}c & +2(2m+3)\mathfrak{C}c \\ +m\mathfrak{A}c & +m\mathfrak{B}c & +m\mathfrak{C}c \end{array}\right\}=0$$

unde utrique coëfficientes sequenti modo determinantur

$$\begin{array}{l|l} B=\dfrac{-mAc}{m(m+1)} & \mathfrak{B}=\dfrac{-(m+2)\mathfrak{A}c}{(m+2)(m+1)} \\ C=\dfrac{-(3m+2)Bc}{2(2m+1)(m+1)} & \mathfrak{C}=\dfrac{-(3m+4)\mathfrak{B}c}{2(2m+3)(m+1)} \\ D=\dfrac{-(5m+4)Cc}{3(3m+2)(m+1)} & \mathfrak{D}=\dfrac{-(5m+6)\mathfrak{C}c}{3(3m+4)(m+1)} \\ E=\dfrac{-(7m+6)Dc}{4(4m+3)(m+1)} & \mathfrak{E}=\dfrac{-(7m+8)\mathfrak{D}c}{4(4m+5)(m+1)} \\ \quad\text{etc.} & \quad\text{etc.} \end{array}$$

ubi bini coëfficientes A et $\mathfrak{A}$ manent indeterminati, ita ut hoc integrale completum sit censendum.

Aliter. Sumta serie, in qua exponentes ipsius x decrescant, fieri debet

$$2\lambda+m=0 \quad \text{seu} \quad m=-2\lambda,$$

ut aequatio nostra sit

$$\partial\partial y - ccx^{-4\lambda}y\partial x^2=0,$$

quae posito

$$p = c\,x^{-2\lambda} \text{ et } y = e^{\int p \partial x} z = e^{\frac{-c}{2\lambda-1} x^{-2\lambda+1}} z,$$

abit in

$$\partial\partial z + 2c\,x^{-2\lambda}\partial x\partial z - 2\lambda c\,x^{-2\lambda-1} z\,\partial x^2 = 0.$$

Ponamus ergo

$$z = A\,x^{\lambda} + B\,x^{3\lambda-1} + C\,x^{5\lambda-2} + D\,x^{7\lambda-3} + \text{etc.}$$

et substitutione facta prodit

$$\left.\begin{array}{lllll} & \lambda(\lambda-1)Ax^{\lambda-2} + (3\lambda-1)(3\lambda-2)Bx^{3\lambda-3} + (5\lambda-2)(5\lambda-3)Cx^{5\lambda-5} & & & \\ +2\lambda Ac\,x^{-\lambda-1} & +2(3\lambda-1)Bc & +2(5\lambda-2)Cc & + & +2(7\lambda-3)Dc \\ -2\lambda Ac & -2\lambda Bc & -2\lambda Cc & & -2\lambda Dc \end{array}\right\} = 0$$

unde coëfficientes ita determinantur

$$B = \frac{-\lambda(\lambda-1)A}{(4\lambda-2)c} = \frac{-\lambda(\lambda-1)A}{2(2\lambda-1)c}$$

$$C = \frac{-(3\lambda-1)(3\lambda-2)B}{(8\lambda-4)c} = \frac{-(3\lambda-1)(3\lambda-2)B}{4(2\lambda-1)c}$$

$$D = \frac{-(5\lambda-2)(5\lambda-3)C}{(12\lambda-6)c} = \frac{-(5\lambda-2)(5\lambda-3)C}{6(2\lambda-1)c}$$

$$E = \frac{-(7\lambda-3)(7\lambda-4)D}{(16\lambda-8)c} = \frac{-(7\lambda-3)(7\lambda-4)D}{8(2\lambda-1)c}$$

etc.

Hic unius tantum litterae A valor arbitrio nostro relinquitur, ex quo haec series tantum integrale particulare exhibet.

Corollarium 1.

941. Ex solutione priori patet, alteram seriem terminari quoties

$$(2i+1)m + 2i = 0, \text{ seu } m = \frac{-2i}{2i+1}$$

alteram vero quoties

$$(2i-1)m + 2i = 0, \text{ seu } m = \frac{-2i}{2i-1}$$

denotante i numerum integrum quemcunque. His ergo casibus integrale saltem particulare finite exprimi potest.

Corollarium 2.

942. Altera solutio praebet seriem finitam, quoties fuerit vel

$$(2i+1)\lambda - i = 0 \quad \text{vel} \quad (2i-1)\lambda - i = 0,$$

hoc est

$$\lambda = \frac{i}{2i \pm 1} \quad \text{et} \quad m = \frac{-2i}{2i \pm 1},$$

ut ante. Reliquis vero casibus haec series in infinitum excurrit.

Corollarium 3.

943. Casus ergo, quibus haec aequatio

$$\partial\partial y - ccx^n \partial x^2 = 0$$

atque adeo posito $y = e^{\int u \partial x}$ etiam haec

$$\partial u + uu\partial x = ccx^n \partial x$$

integrationem saltem particularem admittit, sunt $n = \frac{-4i}{2i \pm 1}$ sumendo pro i numerum integrum quemcunque.

Scholion.

944. Sufficit autem integrale particulare invenisse, cum ex eo facile integrale completum erui possit. Cum enim in integrali insit littera c, dum aequatio differentialis tantum quadratum cc continet, perinde est sive in integrali sumatur $+c$ sive $-c$. Hinc si integrale particulare sit $y = P + cQ$, erit etiam $y = P - cQ$ integrale particulare, unde integrale completum erit

$$y = \alpha(P + cQ) + \beta(P - cQ), \text{ seu } y = \alpha P \mp \beta c Q.$$

Quo haec clarius explicentur, ad solutionem alteram aequationis $\partial\partial y - ccx^{-4\lambda} y \partial x^2 = 0$ accommodentur, pro qua ponendo brevitatis gratia $\frac{1}{1-2\lambda} x^{1-2\lambda} = t$, fecimus $y = e^{ct} z$, et invenimus

$$z = Ax^\lambda - \frac{\lambda(\lambda-1)A}{2(2\lambda-1)c} x^{3\lambda-1} + \frac{\lambda(\lambda-1)(3\lambda-1)(3\lambda-2)A}{2.4(2\lambda-1)^2 cc} x^{5\lambda-2}$$
$$- \frac{\lambda(\lambda-1)(3\lambda-1)(3\lambda-2)(5\lambda-2)(5\lambda-3)A}{2.4.6(2\lambda-1)^3 c^3} x^{7\lambda-3} + \text{etc.}$$

Pro qua expressione distinguendo terminos per potestates pares ipsius c divisos ab iis, qui per potestates impares sunt divisi, scribamus $z = P - cQ$, ita ut jam P et Q tantam potestates pares ipsius c contineant, eritque integrale particulare unum

$$y = e^{ct}(P - cQ)$$

et alterum

$$y = e^{-ct}(P + cQ),$$

unde completum erit

$$y = \tfrac{1}{2}P(\alpha e^{ct} + \beta e^{-ct}) - \tfrac{1}{2}cQ(\alpha e^{ct} - \beta e^{-ct}),$$

Hinc si c sit numerus imaginarius seu $cc = -bb$, ut aequatio sit

$$\partial\partial y + bbx^{-4\lambda}y\partial x^2 = 0, \text{ erit}$$

$$z = P - bQ\sqrt{-1} \text{ et}$$

$$e^{ct} = e^{bt\sqrt{-1}} = \cos. bt + \sqrt{-1}.\sin. bt, \text{ ergo}$$

$$y = P\left(\tfrac{\alpha+\beta}{2}\cos. bt + \tfrac{\alpha-\beta}{2}\sqrt{-1}.\sin. bt\right)$$
$$- bQ\left(\tfrac{\alpha-\beta}{2}\cos. bt + \tfrac{\alpha+\beta}{2}\sqrt{-1}.\sin. bt\right)\sqrt{-1}.$$

Sit $\frac{\alpha+\beta}{2} = \gamma$ et $\frac{\alpha-\beta}{2}\sqrt{-1} = \delta$,

atque integrale completum hoc casu ita exprimetur

$$y = P(\gamma\cos. bt + \delta\sin. bt) - bQ(\delta\cos. bt - \gamma\sin. bt), \text{ seu}$$
$$y = (\gamma P - \delta bQ)\cos. bt + (\delta P + \gamma bQ)\sin. bt.$$

Casus ergo hoc modo integrabiles evolvamus.

Exemplum 1.

945. *Integrale aequationis $\partial\partial y - ccy\partial x^2 = 0$ invenire.*

Hic est $\lambda = 0$ et $z = A$, atque $t = x$, unde ob $P = A$ et $Q = 0$, erit integrale completum

$$y = \alpha e^{cx} + \beta e^{-cx}.$$

Casu autem $cc = -bb$, aequationis $\partial\partial y + bby\partial x^2 = 0$ integrale completum erit

$$y = \gamma \cos. b x + \delta \sin. b x.$$

Exemplum 2.

946. *Integrale aequationis $\partial\partial y - c c x^{-4} y \partial x^2 = 0$ invenire.*

Hic ob $\lambda = 1$ est $z = A x$, et $t = -\frac{1}{x}$, unde ob $P = x$ et $Q = 0$, fit

$$y = (\alpha e^{ct} + \beta e^{-ct}) x.$$

Casu autem $cc = -bb$ aequationis

$$\partial\partial y + bbx^{-4} y \partial x^2 = 0$$

integrale est

$$y = (\alpha \cos. bt + \beta \sin. bt) x, \text{ existente } t = -\frac{1}{x}.$$

Exemplum 3.

947. *Integrale aequationis $\partial\partial y - ccx^{-\frac{4}{3}} y \partial x^2 = 0$ invenire.*

Ob $\lambda = \frac{1}{3}$, fit $B = -\frac{A}{3c}$, et $z = Ax^{\frac{1}{3}} - \frac{A}{3c}$, atque $t = 3x^{\frac{1}{3}}$, unde $P = x^{\frac{1}{3}}$ et $Q = \frac{1}{3cc}$. Integrale ergo erit

$$y = (\alpha e^{ct} + \beta e^{-ct}) x^{\frac{1}{3}} - (\alpha e^{ct} - \beta e^{-ct}) \frac{1}{8c}.$$

Casu autem $cc = -bb$, aequationis

$$\partial\partial y + bbx^{-\frac{4}{3}} y \partial x^2 = 0$$

integrale est

$$y = (\alpha \cos. bt + \beta \sin. bt) x^{\frac{1}{3}} + \frac{1}{3b} (\beta \cos. bt - \alpha \sin. bt).$$

Exemplum 4.

948. *Integrale aequationis* $\partial\partial y - ccx^{-\frac{8}{3}} y \partial x^2 = 0$ *invenire.*

Ob $\lambda = \frac{2}{3}$, fit $B = \frac{A}{3c}$ et $z = Ax^{\frac{2}{3}} + \frac{A}{3c}x$, ita ut sit $P = x^{\frac{2}{3}}$ et $Q = \frac{-x}{3cc}$. Posito ergo $t = -3x^{-\frac{1}{3}}$, integrale ita exprimitur

$$y = x^{\frac{2}{3}}(\alpha e^{ct} + \beta e^{-ct}) + \frac{x}{3c}(\alpha e^{ct} - \beta e^{-ct}).$$

Casu autem $cc = -bb$ aequationis

$$\partial\partial y + bbx^{-\frac{8}{3}} y \partial x^2 = 0$$

integrale est

$$y = x^{\frac{2}{3}}(\alpha \cos. bt + \beta \sin. bt) - \frac{x}{3b}(\beta \cos. bt - \alpha \sin. bt).$$

Exemplum 5.

949. *Integrale aequationis* $\partial\partial y - ccx^{-\frac{8}{3}} y \partial x^2 = 0$ *invenire.*

Ob $\lambda = \frac{2}{5}$, est $B = \frac{-3A}{5c}$ et $C = \frac{+3A}{5^2cc}$, hinc

$$z = Ax^{\frac{2}{5}} - \frac{3A}{3c}x^{\frac{1}{5}} + \frac{3A}{5^2cc}, \text{ ideoque}$$

$$P = x^{\frac{2}{5}} + \frac{3}{5^2cc} \text{ et } Q = \frac{3}{5cc}x^{\frac{1}{5}}.$$

Posito ergo $t = 5x^{\frac{1}{5}}$, integrale erit

$$y = (x^{\frac{2}{5}} + \frac{3}{5^2cc})(\alpha e^{ct} + \beta e^{-ct}) - \frac{3}{5c}x^{\frac{1}{5}}(\alpha e^{ct} - \beta e^{-ct})$$

casu autem $cc = -bb$, aequationis

$$\partial\partial y + bbx^{-\frac{8}{5}} y\partial x^2 = 0$$

integrale est

$$y = (x^{\frac{2}{5}} - \tfrac{3}{5^2 bb})(\alpha\cos. bt + \beta\sin. bt) + \tfrac{3}{5b} x^{\frac{1}{5}}(\beta\cos. bt - \alpha\sin. bt).$$

Problema 119.

950. Aequationis differentio-differentialis

$$\partial\partial y + ccx^{\frac{-4i}{2i-1}} y\partial x^2 = 0$$

integrale completum assignare, denotante i numerum integrum quemcunque.

Solutio.

Sit brevitatis gratia $t = -(2i-1)x^{\frac{-1}{2i-1}}$ unde fit $x^{\frac{1}{2i-1}} = -\frac{(2i-1)}{t}$, ac posito $y = e^{ct}z$, pro valore ipsius z per seriem invento

$$z = Ax^{\frac{i}{2i-1}} + Bx^{\frac{i+1}{2i-1}} + Cx^{\frac{i+2}{2i-1}} + Dx^{\frac{i+3}{2i-1}} + \text{etc.}$$

ob $\lambda = \frac{+i}{2i-1}$, hi coëfficientes ita determinantur

$$B = \frac{i(i-1)A}{2(2i-1)c},\ C = \frac{(i+1)(i-2)B}{4(2i-1)c},\ D = \frac{(i+2)(i-3)C}{6(2i-1)c}\ \text{etc.}$$

quibus substitutis, et introducto valore $x^{\frac{1}{2i-1}} = \frac{-(2i-1)}{t}$, erit

$$z = Ax^{\frac{i}{2i-1}}\left(1 - \frac{i(i-1)}{2ct} + \frac{i(ii-1)(i-2)}{2.4cctt} - \frac{i(ii-1)(ii-4)(i-3)}{2.4.6c^3t^3} + \text{etc.}\right.$$

sive hoc modo

$$z = \frac{A}{t^i}\left(1 - \frac{i(i-1)}{2ct} + \frac{i(ii-1)(i-2)}{2.4c.tt} - \frac{i(ii-1)(ii-4)(i-3)}{2.4.6.c^3t^3} + \text{etc.}\right).$$

Atque hinc aequationis propositae integrale completum ita exprimetur

$$y = t^{-i}\left(1 + \frac{i(ii-1)(i-2)}{2.4cctt} + \frac{i(ii-1)(ii-4)(ii-9)(i-4)}{2.4.6.8c^4t^4} + \text{etc.}\right)(\alpha e^{ct} + \beta e^{-ct})$$
$$- t^{-i}\left(\frac{i(i-1)}{2ct} + \frac{i(ii-1)(ii-4)(i-3)}{2.4.6c^3t^3} + \text{etc.}\right)(\alpha e^{ct} - \beta e^{-ct})$$

ubi in utraque progressione lex formationis singulorum terminorum est manifesta.

Corollarium 1.

951. Hinc quoque illius aequationis

$$\partial\partial y + bb\, x^{\frac{-4i}{2i-1}}\, y\,\partial x^2 = 0$$

integrale completum, manente $t = -(2i-1)\, x^{\frac{-1}{2i-1}}$, est

$$y = t^{-i}\left(1 - \frac{i(ii-1)(i-2)}{2.4bbtt} + \frac{i(ii-1)(ii-4)(ii-9)(i-4)}{2.4.6.8b^4t^4} - \text{etc}\right)(\alpha\cos.bt + \beta\sin.bt)$$
$$+ t^{-i}\left(\frac{i(i-1)}{2bt} - \frac{i(ii-1)(ii-4)(i-3)}{2.4.6b^3t^3} + \frac{i(ii-1)(ii-4)(ii-9)(ii-16)(i-5)}{2.4.6.8.10b^5t^5} - \text{etc.}\right)(\beta\cos.bt - \alpha\sin.bt)$$

Corollarium 2.

952. Si i sit numerus negativus, haec integratio perinde succedit, aequationis enim

$$\partial\partial y - ce\, x^{\frac{-4i}{2i+1}}\, y\,\partial x^2 = 0$$

posito $t = (2i+1)\, x^{\frac{1}{2i+1}}$, integrale erit

$$y = t^{i}\left(1 + \frac{i(ii-1)(i+2)}{2.4cctt} + \frac{i(ii-1)(ii-4)(ii-9)(i+4)}{2.4.6.8c^4t^4} + \text{etc.}\right)(\alpha e^{ct} + \beta e^{-ct})$$
$$- t^{i}\left(\frac{i(i+1)}{2ct} + \frac{i(ii-1)(ii-4)(i+3)}{2.4.6c^3t^3} + \frac{i(ii-1)(ii-4)(ii-9)(ii-16)(i+5)}{2.4.6.8.10c^5t^5} + \text{etc.}\right)(\alpha e^{ct} - \beta e^{-ct}).$$

Corollarium 3.

953. Simili modo hujus aequationis

$$\partial\partial y + bb\, x\, b^{\frac{-4i}{2i+1}}\, y\,\partial x^2 = 0,$$

posito $t = (2i + 1)\, x^{\frac{1}{2i+1}}$, integrale completum erit

$$y = t^i \left(1 - \frac{i(ii-1)(i+2)}{2.4\, bbtt} + \frac{i(ii-1)(ii-4)(ii-9)(i+4)}{2.4.6.8.\, b^4 t^4} - \text{etc}\right)(\alpha \cos. bt + \beta \sin. bt)$$
$$+ t^i \left(\frac{i(i+1)}{2bt} - \frac{i(ii-1)(ii-4)(i+3)}{2.4.6\, b^3 t^3} + \frac{i(ii-1)(ii-4)(ii-9)(ii-16)(i+5)}{2.4.6.8.10\, b^5 t^5} - \text{etc.}\right)(\beta \cos. bt - \alpha \sin. bt).$$

Corollarium 4.

954. In formulis sinus et cosinus continentibus si ponatur

$$\alpha = C \sin. \zeta \text{ et } \beta = C \cos. \zeta,$$

expressiones nostrae ita contrahuntur, ut fiat

$$\alpha \cos. bt + \beta \sin. bt = C \sin. (bt + \zeta) \text{ et}$$
$$\beta \cos. bt - \alpha \sin. bt = C \cos. (bt + \zeta),$$

ut jam hic C et ζ sint constantes arbitrariae integrale completum reddentes.

Scholion.

955. Hinc egregium adipiscimur adminiculum ad casus integrabilitatis hujus aequationis differentialis primi gradus

$$\partial u + uu\partial x + a x^n \partial x = 0$$

agnoscendos, simulque integralia completa definienda; nascitur enim haec aequatio ex ista

$$\partial\partial y + a x^n y \partial x^2 = 0, \text{ ponendo } z = e^{\int u \partial x},$$

unde ex illa vicissim haec oritur, ponendo $u = \frac{\partial y}{y \partial x}$. Cum igitur istius integrale assignare licuerit casibus quibus exponens $n = \frac{-4i}{2i \pm 1}$ iisdem casibus integrale aequationis differentialis primi gradus assignare licebit, ubi quidem duos casus evolvi convenit, prout a fuerit vel numerus negativus $a = -cc$ vel positivus $a = +bb$. Hos igitur duos casus pertractasse operae erit pretium.

Problema 120.

956. Denotante i numerum integrum sive positivum sive negativum quemcunque, invenire integrale hujus aequationis

$$\partial u + uu\partial x - cc\, x^{\frac{-4i}{2i+1}} \partial x = 0.$$

Solutio.

Posito $u = \frac{\partial y}{y \partial x}$, haec aequatio transformatur in istam

$$\partial\partial y - cc\, x^{\frac{-4i}{2i+1}} y \partial x^2 = 0,$$

sumto elemento ∂x constante, cujus integrale assignavimus.

Posito scilicet $t = (2i+1) x^{\frac{1}{2i+1}}$, est

$$y = (\alpha e^{ct} + \beta e^{-ct}) \left(t^i + \frac{i(ii-1)(i+2)}{2.4cc} t^{i-2} + \frac{i(ii-1)(ii-4)(ii-9)(i+4)}{2.4.6.8c^4} t^{i-4} + \text{etc.}\right)$$
$$- (\alpha e^{ct} - \beta e^{-ct}) \left(\frac{i(i+1)}{2c} t^{i-1} + \frac{i(ii-1)(ii-4)(i+3)}{2.4.6c^3} t^{i-3} + \frac{i(ii-1)(ii-4)(ii-9)(ii-16)(i+5)}{2.4.6.8.10c^5} t^{i-5} + \text{etc.}\right)$$

Ponamus brevitatis gratia

$$y = (\alpha e^{ct} + \beta e^{-ct}) P - (\alpha e^{ct} - \beta e^{-ct}) Q,$$

et cum sit

$$\partial t = x^{\frac{-2i}{2i+1}} \partial x, \text{ seu } \partial x = x^{\frac{2i}{2i+1}} \partial t,$$

erit

$$\frac{\partial y}{\partial x} = \frac{(\alpha e^{ct} + \beta e^{-ct})(\partial P - cQ\partial t) + (\alpha e^{ct} - \beta e^{-ct})(cP\partial t - \partial Q)}{x^{\frac{2i}{2i+1}} \partial t},$$

sive

$$\frac{\partial y}{\partial x} = \frac{\alpha e^{ct}(\partial P + cP\partial t - \partial Q - cQ\partial t) + \beta e^{-ct}(\partial P - cP\partial t + \partial Q - cQ\partial t)}{x^{\frac{2i}{2i+1}} \partial t}.$$

At vero est

$$\frac{\partial P}{\partial t} = i\,t^{i-1} + \frac{i(ii-1)(ii-4)}{2.4\,c^2}\,t^{i-3} + \frac{i(ii-1)(ii-4)(ii-9)(ii-16)}{2.4.6.8.\,c^4}\,t^{i-5} + \text{etc.}$$

$$cQ = \frac{i(i+1)}{2}\,t^{i-1} + \frac{i(ii-1)(ii-4)(i+3)}{2.4.6\,c^2}\,t^{i-3} + \frac{i(ii-1)(ii-4)(ii-9)(ii-16)(i+5)}{2.4.6.8.10\,c^4}\,t^{i-5} + \text{etc.}$$

$$cP = c\,t^i + \frac{i(ii-1)(i+2)}{2.4\,c}\,t^{i-2} + \frac{i(ii-1)(ii-4)(ii-9)(i+4)}{2.4.6.8\,c^3}\,t^{i-4} + \text{etc.}$$

$$\frac{\partial Q}{\partial t} = \frac{i(ii-1)}{2\,c}\,t^{i-2} + \frac{i(ii-1)(ii-4)(ii-9)}{2.4.6\,c^3}\,t^{i-4} + \text{etc.}$$

unde colligitur

$$\frac{\partial P - cQ\partial t}{\partial t} = -\frac{i(i-1)}{2}\,t^{i-1} - \frac{i(ii-1)(ii-4)(i-3)}{2.4.6\,c^2}\,t^{i-3} - \frac{i(ii-1)(ii-4)(ii-9)(ii-16)(i-5)}{2.4.6.8.10\,c^4}\,t^{i-5} - \text{etc}$$

$$\frac{cP\partial t - \partial Q}{\partial t} = c\,t^i + \frac{i(ii-1)(i-2)}{2.4\,c}\,t^{i-2} + \frac{i(ii-1)(ii-4)(ii-9)(i-4)}{2.4.6.8\,c^3}\,t^{i-4} + \text{etc.}$$

Ponamus ad abbreviandum

$$P = t^i + \frac{i(ii-1)(i+2)}{2.4\,cc}\,t^{i-2} + \frac{i(ii-1)(ii-4)(ii-9)(i+4)}{2.4.6.8\,c^4}\,t^{i-4} + \text{etc.}$$

$$Q = \frac{i(i+1)}{2c}\,t^{i-1} + \frac{i(ii-1)(ii-4)(i+3)}{2.4.6\,c^3}\,t^{i-3} + \frac{i(ii-1)(ii-4)(ii-9)(ii-16)(i+5)}{2.4.6.8.10\,c^5}\,t^{i-5} + \text{etc.}$$

$$R = t^i + \frac{i(ii-1)(i-2)}{2.4\,cc}\,t^{i-2} + \frac{i(ii-1)(ii-4)(ii-9)(i-4)}{2.4.6.8\,c^4}\,t^{i-4} + \text{etc.}$$

$$S = \frac{i(i-1)}{2c}\,t^{i-1} + \frac{i(ii-1)(ii-4)(i-3)}{2.4.6\,c^3}\,t^{i-3} + \frac{i(ii-1)(ii-4)(ii-9)(ii-16)(i-5)}{2.4.6.8\,10\,c^5}\,t^{i-5} + \text{etc.}$$

ut sit

$$\frac{\partial y}{\partial x} = \frac{(\alpha e^{ct} + \beta e^{-ct})(-cS) + (\alpha e^{ct} - \beta e^{-ct})(cR)}{x^{\frac{2i}{2i+1}}}.$$

Quare cum sit $u = \frac{\partial y}{y\,\partial x}$, erit nostrae aequationis integrale completum

$$\tfrac{1}{c}x^{\frac{2i}{2i+1}}u = \frac{(\alpha e^{ct} - \beta e^{-ct})R - (\alpha e^{ct} + \beta e^{-ct})S}{(\alpha e^{ct} + \beta e^{-ct})P - (\alpha e^{ct} - \beta e^{-ct})Q}, \text{ sive}$$

$$\tfrac{1}{c}x^{\frac{2i}{2i+1}}u = \frac{\alpha e^{ct}(R - S) - \beta e^{-ct}(R + S)}{\alpha e^{ct}(P - Q) + \beta e^{-ct}(P + Q)},$$

quod ob rationem constantium $\alpha : \beta$ arbitrariam, est completum.

Corollarium 1.

957. Quaternae formulae P, Q, R, S, quae singulae casibus, quibus i est numerus integer, abrumpuntur, ita a se invicem pendent, ut sit primo

$$R = P - \frac{\partial Q}{c \partial t} \text{ et } S = Q - \frac{\partial P}{c \partial t};$$

tum vero

$$\partial P + \partial R = \frac{+2iR\partial t}{t} \text{ et } \partial Q + \partial S = \frac{2iS\partial t}{t}.$$

Corollarium 2.

958. Posito ergo vel $\alpha = 0$ vel $\beta = 0$, integralia particularia algebraica aequationis

$$\partial u + uu\partial x - cc\, x^{\frac{-4i}{2i+1}} \partial x = 0$$

exhiberi possunt, quae sunt

$$\frac{1}{c} x^{\frac{2i}{2i+1}} u = \frac{R-S}{P-Q} \text{ et } \frac{1}{c} x^{\frac{2i}{2i+1}} u = \frac{-S-R}{P+Q},$$

ideoque hac una formula comprehendi possunt

$$\frac{1}{c} x^{\frac{2i}{2i+1}} u = \frac{-S \pm R}{P \mp Q}.$$

Scholion 1.

959. Pro variis ergo valoribus numeri i tam quantitas t quam litterae P, Q, R, S sequenti modo se habebunt: Primo scilicet si $i = 0$, erit $t = x$, atque $P = 1$, $Q = 0$, $R = 1$ et $S = 0$; reliquos casus in sequenti tabella repraesentemus

$i = -1,\ t = -\frac{1}{x}$	$i = 1,\ t = 3x^{\frac{1}{3}}$
$P = \frac{1}{t},\quad Q = 0$	$P = t,\ Q = \frac{1}{c}$
$R = \frac{1}{t},\quad S = \frac{1}{ctt}$	$R = t,\ S = 0$

$i = -2,\ t = -\frac{3}{x^{\frac{1}{3}}}$ $P = \frac{1}{t^2},\ Q = \frac{1}{c\,t^3}$ $R = \frac{1}{t^2} + \frac{3}{cc\,t^4},\ S = \frac{3}{c\,t^3},$	$i = 2,\ t = 5\,x^{\frac{1}{5}}$ $P = t^2 + \frac{3}{cc},\ Q = \frac{3}{c}\,t$ $R = t^2,\ S = \frac{1}{c}\,t.$
$i = -3,\ t = \frac{-5}{x^{\frac{1}{5}}}$ $P = \frac{1}{t^3} + \frac{1.3}{cc\,t^5}$ $Q = \frac{3}{c\,t^4}$ $R = \frac{1}{t^3} + \frac{3.5}{cc\,t^5}$ $S = \frac{6}{c\,t^4} + \frac{3.5}{c^3\,t^6}$	$i = 3,\ t = 7\,x^{\frac{1}{7}}$ $P = t^3 + \frac{3.5}{cc}\,t$ $Q = \frac{6}{c}\,tt + \frac{3.5}{c^3}$ $R = t^3 + \frac{1.3}{cc}\,t$ $S = \frac{3}{c}\,tt$
$i = -4,\ t = -\frac{7}{x^{\frac{1}{7}}}$ $P = \frac{1}{t^4} + \frac{3.5}{cc\,t^6}$ $Q = \frac{6}{c\,t^5} + \frac{3.5}{c^3\,t^7}$ $R = \frac{1}{t^4} + \frac{3.3.5}{cc\,t^6} + \frac{3.5.7}{c^4\,t^8}$ $S = \frac{10}{c\,t^5} + \frac{3.5.7}{c^3\,t^7}$	$i = 4,\ t = 9\,x^{\frac{1}{9}}$ $P = t^4 + \frac{3.3.5}{cc}\,t^2 + \frac{3.5.7}{c^4}$ $Q = \frac{10}{c}\,t^3 + \frac{3.5.7}{c^3}\,t$ $R = t^4 + \frac{3.5}{cc}\,t^2$ $S = \frac{6}{c}\,t^3 + \frac{3.5}{c^3}\,t$
$i = -5,\ t = \frac{-9}{x^{\frac{1}{9}}}$ $P = \frac{1}{t^5} + \frac{3.3.5}{cc\,t^7} + \frac{3.5.7}{c^4\,t^9}$ $Q = \frac{10}{c\,t^6} + \frac{3.5.7}{c^3\,t^8}$ $R = \frac{1}{t^5} + \frac{3.5.7}{cc\,t^7} + \frac{3.5.7.9}{c^4\,t^9}$ $S = \frac{15}{c\,t^6} + \frac{4.3.5.7}{c^3\,t^8} + \frac{3.5.7.9}{c^5\,t^{10}}$	$i = 5,\ t = 11\,x^{\frac{1}{11}}$ $P = t^5 + \frac{3.5.7}{cc}\,t^3 + \frac{3.5.7.9}{c^4}\,t$ $Q = \frac{15}{c}\,t^4 + \frac{4.3.5.7}{c^3}\,t^2 + \frac{3.5.7.9}{c^5}$ $R = t^5 + \frac{3.3.5}{cc}\,t^3 + \frac{3.5.7}{c^4}\,t$ $S = \frac{10}{c}\,t^4 + \frac{3.5.7}{c^3}\,t^2$

$$i = -6,\ t = \frac{-11}{x^{\frac{1}{11}}}$$

$$P = \frac{1}{t^6} + \frac{3.5.7}{cct^8} + \frac{3.5.7.9}{c^4t^{10}}$$

$$Q = \frac{15}{ct^7} + \frac{4.3.5.7}{c^3t^9} + \frac{3.5.7.9}{c^5t^{11}}$$

$$R = \frac{1}{t^6} + \frac{2.3.5.7}{cct^8} + \frac{5.3.5.7.9}{c^4t^{10}} + \frac{3.5.7.9.11}{c^6t^{12}}$$

$$S = \frac{21}{ct^7} + \frac{4.5.7.9}{c^3t^9} + \frac{3.5.7.9.11}{c^5t^{11}}.$$

$$i = 6,\ t = 13x^{\frac{1}{13}}$$

$$P = t^6 + \frac{2.3.5.7}{cc}t^4 + \frac{5.3.5.7.9}{c^4}t^2 + \frac{3.5.7.9.11}{c^6}$$

$$Q = \frac{21}{c}t^5 + \frac{4.5.7.9}{c^3}t^3 + \frac{3.5.7.9.11}{c^5}t$$

$$R = t^6 + \frac{3.5.7}{cc}t^4 + \frac{3.5.7.9}{c^4}t^2$$

$$S = \frac{15}{c}t^5 + \frac{4.3.5.7}{c^3}t^3 + \frac{3.5.7.9}{c^5}t.$$

Scholion. 2.

960. Dum hae formulae diligentius considerantur, nova se prodit relatio inter valores litterarum P, Q, R, S, quae in hoc consistit, ut perpetuo sit $PR - QS = t^{2i}$, cujus veritas primo quidem per inductionem deprehenditur, tum vero etiam per relationes supra datas demonstrari potest. Si enim valores

$$R = P - \frac{\partial Q}{c\partial t} \quad \text{et} \quad S = Q - \frac{\partial P}{c\partial t}$$

in aequationibus

$$\partial P + \partial R = \frac{2iR\partial t}{t} \quad \text{et} \quad \partial Q + \partial S = \frac{2iS\partial t}{t}$$

substituantur, oriuntur hae duae aequationes

$$2\partial P - \frac{\partial\partial Q}{c\partial t} = \frac{2iP\partial t}{t} - \frac{2i\partial Q}{ct} \quad \text{et}$$

$$2\partial Q - \frac{\partial\partial P}{c\partial t} = \frac{2iQ\partial t}{t} - \frac{2i\partial P}{ct},$$

quarum illa per P, haec vero per — Q multiplicata junctim dant

$$2P\partial P - 2Q\partial Q + \frac{Q\partial\partial P - P\partial\partial Q}{c\partial t} = \frac{2i\partial t}{t}(PP - QQ)$$
$$+ \frac{2i}{ct}(Q\partial P - P\partial Q).$$

Ponatur

$$PP - QQ = M \text{ et } \frac{Q\partial P - P\partial Q}{c\partial t} = N, \text{ erit}$$
$$\partial M + \partial N = \frac{2i\partial t}{t}(M + N), \text{ seu } \frac{\partial M + \partial N}{M + N} = \frac{2i\partial t}{t}$$

hincque integrando $M + N = C\,t^{2i}$. At est

$$M + N = P\left(P - \frac{\partial Q}{c\partial t}\right) - Q\left(Q - \frac{\partial P}{c\partial t}\right) = PR - QS;$$

evidens autem est pro constante C unitatem accipi debere.

Problema 121.

961. Denotante i numerum integrum sive positivum sive negativum quemcunque, invenire integrale completum hujus aequationis

$$\partial u + uu\partial x + bb\,x^{\frac{-4i}{2i+1}}\partial x = 0.$$

Solutio.

Posito $u = \frac{\partial y}{y\partial x}$, haec aequatio transformatur in istam

$$\partial\partial y + bb\,x^{\frac{-4i}{2i+1}}y\,\partial x^2 = 0,$$

sumto elemento ∂x constante, cujus integrale supra est assignatum. Scilicet posito $t = (2i + 1)\,x^{\frac{1}{2i+1}}$, invenimus (953, 954)

$$y = C\left(t^i - \frac{i(ii-1)(i+2)}{2.4\,bb}t^{i-2} + \frac{i(ii-1)(ii-4)(ii-9)(i+4)}{2.4.6.8\,b^4}t^{i-4} - \text{etc.}\right)\sin.(bt + \zeta)$$
$$+ C\left(\frac{i(i+1)}{2b}t^{i-1} - \frac{i(ii-1)(ii-4)(i+3)}{2.4.6.\,b^3}t^{i-3} + \text{etc.}\right)\cos.(bt + \zeta),$$

cujus loco brevitatis gratia scribamus

$$y = C\,P\sin.(bt + \zeta) + C\,Q\cos.(bt + \zeta).$$

Hinc ob

$$\partial t = x^{\frac{-2i}{2i+1}} \partial x \quad \text{seu} \quad \partial x = x^{\frac{2i}{2i+1}} \partial t,$$

erit

$$\frac{\partial y}{\partial x} = \frac{C(\partial P - bQ\partial t)\sin.(bt+\zeta) + C(\partial Q + bP\partial t)\cos.(bt+\zeta)}{x^{\frac{2i}{2i+1}}\partial t}:$$

unde cum sit $u = \frac{\partial y}{y\partial x}$, erit

$$u = \frac{(\partial P - bQ\partial t)\sin.(bt+\zeta) + (\partial Q + bP\partial t)\cos.(bt+\zeta)}{x^{\frac{2i}{2i+1}}\partial t\,[P\sin.(bt+\zeta) + Q\cos.(bt+\zeta)]}.$$

Ponamus

$$P + \frac{\partial Q}{b\partial t} = R \quad \text{et} \quad Q - \frac{\partial P}{b\partial t} = S,$$

ut sit

$$\frac{1}{b} x^{\frac{2i}{2i+1}} u = \frac{R\cos.(bt+\zeta) - S\sin.(bt+\zeta)}{P\sin.(bt+\zeta) + Q\cos.(bt+\zeta)},$$

erit

$$P = t^i - \frac{i(ii-1)(i+2)}{2.4\,bb} t^{i-2} + \frac{i(ii-1)(ii-4)(ii-9)(i+4)}{2.4.6.8.10\,b^4} t^{i-4} - \text{etc.}$$

$$Q = \frac{i(i+1)}{2b} t^{i-1} - \frac{i(ii-1)(ii-4)(i+3)}{2.4.6\,b^3} t^{i-3} + \frac{i(ii-1)(ii-4)(ii-9)(ii-16)(i+5)}{2.4.6.8.10\,b^5} t^{i-5} - \text{etc.}$$

$$R = t^i - \frac{i(ii-1)(i-2)}{2.4.\,bb} t^{i-2} + \frac{i(ii-1)(ii-4)(ii-9)(i-4)}{2.4.6.8\,b^4} t^{i-4} - \text{etc.}$$

$$S = \frac{i(i-1)}{2b} t^{i-1} - \frac{i(ii-1)(ii-4)(i-3)}{2.4.6\,b^3} t^{i-3} + \frac{i(ii-1)(ii-4)(ii-9)(ii-16)(i-5)}{2.4.6.8.10\,b^5} t^{i-5} - \text{etc.}$$

atque ob angulum ζ introductum hoc integrale erit completum.

Corollarium 1.

962. Quaternarum ergo litterarum P, Q, R, S valores ita a se invicem pendent, ut sit primo

$$R = P + \frac{\partial Q}{b\partial t} \quad \text{et} \quad S = Q - \frac{\partial P}{b\partial t}$$

tum vero etiam patet fore

$$\partial P + \partial R = \frac{2iR\partial t}{t} \quad \text{et} \quad \partial Q + \partial S = \frac{2iS\partial t}{t}.$$

Corollarium 2.

963. Deinde etiam colligitur fore $PR + QS = t^{2i}$ quae aequalitas ex praecedentis problematis formulis deducitur, sumto $cc = -bb$, ubi Q et S abeunt in $Q\sqrt{-1}$ et $S\sqrt{-1}$.

Corollarium 3.

964. Hic casus a praecedente etiam hoc differt, quod hic nulla dentur integralia particularia algebraica. Quicunque enim valor angulo constanti ζ tribuatur, integrale semper sinum et cosinum cujusdam anguli involvit.

Scholion 1.

965. Cum igitur aequationis

$$\partial u + uu\partial x + bb x^{\frac{-4i}{2i+1}}\partial x = 0$$

integrale completum, posito $t = (2i+1)x^{\frac{1}{2i+1}}$, sit

$$\frac{1}{b}x^{\frac{2i}{2i+1}}u = \frac{R\cos.(bt+\zeta) - S\sin.(bt+\zeta)}{P\sin.(bt+\zeta) + Q\cos.(bt+\zeta)};$$

pro singulis valoribus numeri i quantitas t cum litteris P, Q, R, S ita se habebit: Primo si $i = 0$, erit $P = 1$, $Q = 0$, $R = 1$ et $S = 0$, item $t = x$, ita ut integrale sit $\frac{1}{b}u = \frac{\cos.(bx+\zeta)}{\sin.(bx+\zeta)}$; reliquos casus sequens tabella exhibet:

$i = -1,\ t = -\frac{1}{x}$	$i = 1,\ t = 3x^{\frac{1}{3}}$
$P = \frac{1}{t},\quad Q = 0$	$P = t,\ Q = \frac{1}{b}$
$R = \frac{1}{t},\quad S = \frac{1}{btt}$	$R = t,\ S = 0$

$i = -2,\ t = -\frac{3}{x^{\frac{1}{3}}}$ $P = \frac{1}{t^2},\ Q = \frac{1}{b\,t^3}$ $R = \frac{1}{t^2} + \frac{3}{b\,b\,t^4},\ S = \frac{3}{b\,t^3},$	$i = 2,\ t = 5\,x^{\frac{1}{5}}$ $P = t^2 + \frac{3}{b\,b},\ Q = \frac{3}{b}\,t$ $R = t^2,\ S = \frac{1}{b}\,t.$
$i = -3,\ t = \frac{-5}{x^{\frac{1}{5}}}$ $P = \frac{1}{t^3} + \frac{1\cdot 3}{b\,b\,t^5}$ $Q = \frac{3}{b\,t^4}$ $R = \frac{1}{t^3} + \frac{3\cdot 5}{b\,b\,t^5}$ $S = \frac{6}{b\,t^4} + \frac{3\cdot 5}{b^3\,t^6}$	$i = 3,\ t = 7\,x^{\frac{1}{7}}$ $P = t^3 + \frac{3\cdot 5}{b\,b}\,t$ $Q = \frac{6}{b}\,t\,t + \frac{3\cdot 5}{b^3}$ $R = t^3 + \frac{1\cdot 3}{b\,b}\,t$ $S = \frac{3}{b}\,t\,t$
$i = -4,\ t = -\frac{7}{x^{\frac{1}{7}}}$ $P = \frac{1}{t^4} + \frac{3\cdot 5}{b\,b\,t^6}$ $Q = \frac{6}{b\,t^5} + \frac{3\cdot 5}{b^3\,t^7}$ $R = \frac{1}{t^4} + \frac{3\cdot 3\cdot 5}{b\,b\,t^6} + \frac{3\cdot 5\cdot 7}{b^4\,t^8}$ $S = \frac{10}{b\,t^5} + \frac{3\cdot 5\cdot 7}{b^3\,t^7}$	$i = 4,\ t = 9\,x^{\frac{1}{9}}$ $P = t^4 + \frac{3\cdot 3\cdot 5}{b\,b}\,t^2 + \frac{3\cdot 5\cdot 7}{b^4}$ $Q = \frac{10}{b}\,t^3 + \frac{3\cdot 5\cdot 7}{b^3}\,t$ $R = t^4 + \frac{3\cdot 5}{b\,b}\,t^2$ $S = \frac{6}{b}\,t^3 + \frac{3\cdot 5}{b^3}\,t$
$i = -5,\ t = \frac{-9}{x^{\frac{1}{9}}}$ $P = \frac{1}{t^5} + \frac{3\cdot 3\cdot 5}{b\,b\,t^7} + \frac{3\cdot 5\cdot 7}{b^4\,t^9}$ $Q = \frac{10}{b\,t^6} + \frac{3\cdot 5\cdot 7}{b^3\,t^8}$ $R = \frac{1}{t^5} + \frac{3\cdot 5\cdot 7}{b\,b\,t^7} + \frac{3\cdot 5\cdot 7\cdot 9}{b^4\,t^9}$ $S = \frac{15}{b\,t^6} + \frac{4\cdot 3\cdot 5\cdot 7}{b^3\,t^8} + \frac{3\cdot 5\cdot 7\cdot 9}{b^5\,t^{10}}$	$i = 5,\ t = 11\,x^{\frac{1}{11}}$ $P = t^5 + \frac{3\cdot 5\cdot 7}{b\,b}\,t^3 + \frac{3\cdot 5\cdot 7\cdot 9}{b^4}\,t$ $Q = \frac{15}{b}\,t^4 + \frac{4\cdot 3\cdot 5\cdot 7}{b^3}\,t^2 + \frac{3\cdot 5\cdot 7\cdot 9}{b^5}$ $R = t^5 + \frac{3\cdot 3\cdot 5}{b\,b}\,t^3 + \frac{3\cdot 5\cdot 7}{b^4}\,t$ $S = \frac{10}{b}\,t^4 + \frac{3\cdot 5\cdot 7}{b^3}\,t^2$

$$i = -6,\quad t = \frac{-11}{x^{\frac{1}{11}}}$$

$$P = \frac{1}{t^6} + \frac{3.5.7}{bb\,t^8} + \frac{3.5.7.9}{b^4\,t^{10}}$$

$$Q = \frac{15}{b\,t^7} + \frac{4.3.5.7}{b^3\,t^9} + \frac{3.5.7.9}{b^5\,t^{11}}$$

$$R = \frac{1}{t^6} + \frac{2.3.5.7}{bb\,t^8} + \frac{5.3.5.7.9}{b^4\,t^{10}} + \frac{3.5.7.9.11}{b^6\,t^{12}}$$

$$S = \frac{21}{b\,t^7} + \frac{4.5.7.9}{b^3\,t^9} + \frac{3.5.7.9.11}{b^5\,t^{11}}$$

$$i = 6,\quad t = 13\,x^{\frac{1}{13}}$$

$$P = t^6 + \frac{2.3.5.7}{bb}t^4 + \frac{5.3.5.7.9}{b^4}t^2 + \frac{3.5.7.9.11}{b^6}$$

$$Q = \frac{21}{b}t^5 + \frac{4.5.7.9}{b^3}t^3 + \frac{3.5.7.9.11}{b^5}t$$

$$R = t^6 + \frac{3.5.7}{bb}t^4 + \frac{3.5.7.9}{b^4}t^2$$

$$S = \frac{15}{b}t^5 + \frac{4.3.5.7}{b^3}t^3 + \frac{3.5.7.9}{b^5}t.$$

Scholion 2.

966. Forma integralis inventa modum suppeditat, aequationem propositam

$$\partial u + uu\,\partial x + A\,x^{\frac{-4i}{2i+1}}\,\partial x = 0$$

in speciem simpliciorem transformandi. Primo enim ponatur

$$x^{\frac{2i}{2i+1}}\,u = v,\quad \text{seu}\quad u = x^{\frac{-2i}{2i+1}}\,v,$$

ac prodibit

$$x^{\frac{-2i}{2i+1}}\,\partial v - \frac{2i}{2i+1}\,x^{\frac{-4i-1}{2i+1}}\,v\,\partial x + x^{\frac{-4i}{2i+1}}\,vv\,\partial x$$
$$+ A\,x^{\frac{-4i}{2i+1}}\,\partial x = 0,\ \text{seu}$$

$$\partial v - \frac{2i}{2i+1}\cdot\frac{v\,\partial x}{x} + x^{\frac{-2i}{2i+1}}\,vv\,\partial x + A\,x^{\frac{-2i}{2i+1}}\,\partial x = 0.$$

Ponatur porro $t = (2i+1)\, x^{\frac{1}{2i+1}}$, erit

$$\partial t = x^{\frac{-2i}{2i+1}} \partial x, \text{ et } \frac{\partial t}{t} = \frac{1}{2i+1} \cdot \frac{\partial x}{x},$$

unde fit

$$\partial v - \frac{2iv\partial t}{t} + vv\partial t + A\partial t = 0.$$

Sit insuper $v = \frac{i}{t} + z$, ut prodeat

$$-\frac{i\partial t}{tt} + \partial z - \frac{2ii\partial t}{tt} - \frac{2iz\partial t}{t} + \frac{ii\partial t}{tt} + \frac{2iz\partial t}{t} + zz\partial t + A\partial t = 0$$

seu

$$\partial z + zz\partial t - \frac{i(i+1)\partial t}{tt} + A\partial t = 0,$$

quae ergo quoties i est numerus integer, est integrabilis. Simili modo haec aequatio

$$\partial u + uu\partial x + A x^n \partial x = 0$$

generalius ita transformari potest: posito $u = x^\lambda v$ et $v = z - \frac{1}{2}\lambda x^{-\lambda-1}$, obtinetur

$$\partial z + x^\lambda zz\partial x + \tfrac{1}{4}\lambda(\lambda+2)x^{-\lambda-2}\partial x + A x^{n-\lambda}\partial x = 0,$$

quae porro posito $x^\lambda \partial x = \partial t$, seu $x^{\lambda+1} = (\lambda+1)t$, abit in

$$\partial z + zz\partial t + \frac{\lambda(\lambda+2)\partial t}{4(\lambda+1)^2 tt} + A(\lambda+1)^{\frac{n-2\lambda}{\lambda+1}}\, t^{\frac{n-2\lambda}{\lambda+1}}\, \partial t = 0,$$

quae aequatio est integrabilis, quoties $n = \frac{-4i}{2i+1}$, unde numerum λ pro lubitu assumendo, innumerabiles formae exhiberi possunt. Si capiatur $\lambda = -1$, fit $t = lx$ et

$$\partial z + zz\partial t - \tfrac{1}{4}\partial t + A e^{(n+2)t}\partial t = 0.$$

CAPUT VIII.

DE
ALIARUM AEQUATIONUM DIFFERENTIO - DIFFERENTIALIUM RESOLUTIONE PER SERIES INFINITAS.

Problema 122.

967.

Formam generalem aequationum differentio-differentialium, quas commode per series resolvere licet, exhibere, earumque integralia investigare.

Solutio.

Primo alias aequationes commode per series resolvere non licet, nisi in quibus altera variabilis y cum suis differentialibus ∂y et $\partial\partial y$ nusquam plus una dimensione obtinet; quoniam pro y seriem infinitam substituendo in calculos nimis molestos incideremus, si usquam plures dimensiones ingrederentur. Hujusmodi ergo aequationes in hac forma

$$\partial\partial y + M\partial x\partial y + Ny\partial x^2 = X\partial x^2$$

continentur. Tum vero ut seriei pro y assumtae quilibet terminus per solum praecedentem determinetur, qui est casus resolutionis maxime notabilis, duplicis tantum generis terminos ratione alterius variabilis x inesse oportet, siquidem ad dimensiones, quas ipsa x cum suo differentiali ∂x constituit, respiciamus. Unde primo quidem, rejecto termino $X\partial x^2$, aequationes hoc modo resolubiles in hac forma continentur

$$xx(a+bx^n)\partial\partial y+x(c+ex^n)\partial x\partial y+(f+gx^n)y\partial x^2=0.$$

Pro cujus resolutione fingamus

$$y=Ax^\lambda+Bx^{\lambda+n}+Cx^{\lambda+2n}+Dx^{\lambda+3n}+\text{etc.}$$

et facta substitutione, sequens serierum summa ad nihilum redigi debet

$$\begin{array}{llll}
\lambda(\lambda-1)Aax^\lambda & +(\lambda+n)(\lambda+n-1)Bax^{\lambda+n} & +(\lambda+2n)(\lambda+2n-1)Cax^{\lambda+2n} & +\text{ etc.}\\
 & +\lambda(\lambda-1)Ab & +(\lambda+n)(\lambda+n-1)Bb & \\
+\lambda Ac & +(\lambda+n)Bc & +(\lambda+2n)Cc & \\
 & +\lambda Ae & +(\lambda+n)Be & \\
+Af & +Bf & +Cf & \\
 & +Ag & +Bg. &
\end{array}$$

Hic ergo primo exponens λ ita accipi debet, ut sit

$$\lambda(\lambda-1)a+\lambda c+f=0,$$

tum vero pro reliquis fieri oportet

$$[(\lambda+n)(\lambda+n-1)a+(\lambda+n)c+f]B=-[\lambda(\lambda-1)b+\lambda e+g]A,$$

$$[(\lambda+2n)(\lambda+2n-1)a+(\lambda+2n)c+f]C=$$
$$-[(\lambda+n)(\lambda+n-1)b+(\lambda+n)e+g]B,$$

$$[(\lambda+3n)(\lambda+3n-1)a+(\lambda+3n)c+f]D=$$
$$-[(\lambda+2n)(\lambda+2n-1)b+(\lambda+2n)e+g)C$$

etc.

Cum igitur sit

$$\lambda(\lambda-1)a+\lambda c+f=0,$$

si ponamus brevitatis causa

$$(\lambda(\lambda-1)b+\lambda e+g=h,$$

erit

$$[n(n+2\lambda-1)a+nc]B=-hA$$

$$[2n(2n+2\lambda-1)a+2nc]C=-[n(n+2\lambda-1)b+ne+h]B$$

$$[3n(3n+2\lambda-1)a+3nc]D=-[2n(2n+2\lambda-1)b+2ne+h]C.$$

etc.

Quia ergo nisi $a=0$, pro λ gemini inveniuntur valores, scilicet

$$\lambda=\frac{a-c\pm\sqrt{[(a-c)^2-4af]}}{2a},$$

binae series pro y inveniuntur, quae utcunque combinatae integrale completum aequationis propositae praebent.

Aliter

Proposita aequatione hac

$$xx(a+bx^n)\partial\partial y+x(c+ex^n)\partial x\partial y+(f+gx^n)y\partial x^2=0,$$

series quoque ordine retrogrado fingi potest

$$y=Ax^\lambda+Bx^{\lambda-n}+Cx^{\lambda-2n}+Dx^{\lambda-3n}+\text{etc.}$$

unde oritur ad nihilum reducendum

$$\begin{array}{lll}
+\lambda(\lambda-1)Abx^{\lambda+n}+(\lambda-n)(\lambda-n-1)Bbx^\lambda+(\lambda-2n)(\lambda-2n-1)Cbx^{\lambda-n}+\text{etc} \\
\qquad\qquad +\lambda(\lambda-1)Aa \qquad\qquad +(\lambda-n)(\lambda-n-1)Ba \\
+\lambda Ae \qquad\quad +(\lambda-n)Be \qquad\qquad +(\lambda-2n)Ce \\
\qquad\qquad +\lambda Ac \qquad\qquad +(\lambda-n)Bc \\
+Ag \qquad\quad +Bg \qquad\qquad +Cg \\
\qquad\qquad +Af \qquad\qquad +Bf
\end{array}$$

Hic ergo exponentem λ ita accipi oportet, ut fiat

$$\lambda(\lambda-1)b+\lambda e+g=0.$$

Tum vero si ponamus

$$\lambda(\lambda-1)a+\lambda c+f=h,$$

determinatio coëfficientium ita se habebit

$$n[(n - 2\lambda + 1)b - e]B = -hA,$$
$$2n[(2n - 2\lambda + 1)b - e]C = -[n(n - 2\lambda + 1)a - nc + h]B,$$
$$3n[(3n - 2\lambda + 1)b - e]D = -[2n(2n - 2\lambda + 1)a - 2nc + h]C,$$
etc.

Corollarium 1.

968. Ex priore solutione, si i denotet numerum integrum positivum, series assumta alicubi abrumpetur, si fuerit

$$in(in + 2\lambda - 1)b + ine + h = 0, \text{ vel}$$
$$(\lambda + in)(\lambda + in - 1)b + (\lambda + in)e + g = 0,$$

hoc est

$$\left.\begin{matrix} \lambda\lambda b + \lambda(2in - 1)b + in(in - 1)b \\ + \lambda e \qquad\qquad + ine + g \end{matrix}\right\} = 0.$$

Corollarium 2.

969. Aequatio ergo nostra integrationem admittit, si litterae f et g ita fuerint comparatae, ut sit

$$f = -\lambda(\lambda - 1)a - \lambda c \text{ et}$$
$$g = -(\lambda + in)(\lambda + in - 1)b - (\lambda + in)e.$$

Vel sumtis duobus numeris μ et ν, ut sit $\nu - \mu$ divisibile per exponentem n, si fuerit

$$f = -\mu(\mu - 1)a - \mu c \text{ et } g = -\nu)\nu - 1)b - \nu e.$$

Corollarium. 3.

970. Cum hinc sit

$$\mu = \frac{a - c \pm \sqrt{[(a-c)^2 - 4af]}}{2a} \text{ et } \nu = \frac{b - e \pm \sqrt{[(b-e)^2 - 4bg]}}{2b},$$

aequatio habebit integrale algebraicum, si fuerit $\nu - \mu = in$, denotante i numerum integrum positivum: hoc est si sit

$$in = \frac{c}{2a} - \frac{e}{2b} \pm \frac{\sqrt{[(b-e)^2 - 4bg]}}{2b} \mp \frac{\sqrt{[(a-c)^2 - 4af)}}{2a}.$$

Corollarium 4.

971. Pro serie autem invenienda si eveniat, ut exponens λ fiat imaginarius, notari convenit esse

$$x^{\alpha+\beta\sqrt{-1}} = x^{\alpha} . e^{\beta\sqrt{-1}.lx} = x^{\alpha}(\cos.\beta lx + \sqrt{-1}.\sin.\beta lx),$$

unde binae series ita combinari poterunt, ut integrale consequatur formam realem.

Scholion.

972. Utraque solutio generatim spectata duplicem seriem pro variabili y suppeditat, pro gemino exponentis λ valore, quarum combinatio integrale completum exhibet. Solutio scilicet prior pro exponente λ hos duos praebet valores

$$\lambda = \frac{a-c \pm \sqrt{[(a-c)^2 - 4af]}}{2a},$$

solutio vero posterior

$$\lambda = \frac{b-e \pm \sqrt{[(b-e)^2 - 4bg]}}{2b},$$

ita ut hoc modo integrale completum duplici modo exprimi possit; quae binae formae etiamsi maxime diversae, atque adeo interdum altera per exponentes imaginarios progrediatur, dum altera habet reales, tamen sibi aequipollentes esse debent. Quin etiam evenire potest, ut altera solutio vel etiam utraque ad integrale completum exhibendum sit inepta, dum unicam seriem suppeditet. Incommodum hoc pro utraque solutione duplici modo accidere potest, pro priori nempe solutione, ubi exponentem λ ex hac aequatione

$$\lambda(\lambda - 1)a + \lambda c + f = 0,$$

definiri oportet, unicus inde pro λ eruitur valor, si fuerit vel $a = 0$, vel $4af = (a-c)^2$, priori casu tantum fit $\lambda = -\frac{f}{c}$, altero ipsius λ valore quasi in infinitum abeunte. Posteriori casu vero ambo ipsius λ valores fiunt inter se aequales, scilicet $\lambda = \frac{a-c}{2a}$. Idem incommodum in altera solutione locum habet, si fuerit vel $b = 0$,

vel $4bg = (b-e)^2$: unde patet fieri posse, ut altera solutio hujusmodi incommodo laboret, dum altera eo careat, quin etiam ut utraque eodem inquinetur. Quocirca ostendi conveniet, quemadmodum etiam his casibus integrale completum investigari debeat; quorsum etiam casum referamus, quo ambo ipsius λ valores fiunt imaginarii, quandoquidem ad imaginariam speciem tollendam singulari artificio est opus. Denique vero etiam binae series pro y exhibendae difficultate premuntur, quoties bini valores ipsius λ differentiam habent per exponentem n divisibilem, quorum casuum evolutio etiam explicari meretur.

Problema 123.

973. **Proposita aequatione differentio-differentiali**

$$xx(a+bx^n)\partial\partial y + x(c+ex^n)\partial x\partial y + (f+gx^n)y\partial x^2 = 0,$$

si eveniat, ut binae series ascendentes pro y assumtae vel in unam coalescant, vel altera fiat impossibilis, integrale completum per series exprimere.

Solutio.

Assumta serie

$$y = Ax^\lambda + Bx^{\lambda+n} + Cx^{\lambda+2n} + Dx^{\lambda+3n} + \text{etc.}$$

si eveniat ut bini valores ipsius λ ex aequatione

$$\lambda(\lambda-1)a + \lambda c + f = 0$$

vel fiant aequales, vel differentiam per n divisibilem obtineant, valor ipsius y praeter potestates ipsius x etiam logarithmum ipsius x involvet. Quare pro aequationis resolutione statim ponamus $y = u + v\,lkx$, ut sit $y = u + v\,lx + \alpha v$, denotante α quantitatem constantem quamcunque. Hinc erit

$$\partial y = \partial u + \frac{v\partial x}{x} + \partial v\,lkx,$$

$$\partial\partial y = \partial\partial u + \frac{2\partial x\partial v}{x} - \frac{v\partial x^2}{xx} + \partial\partial v\,lkx,$$

quibus valoribus substitutis, aequatio nostra hanc induet formam

$$\left.\begin{array}{r} xx(a+bx^n)\partial\partial u+2x(a+bx^n)\partial x\partial v-(a+bx^n)v\partial x^2 \\ +x(c+ex^n)\partial x\partial u+(c+ex^n)v\partial x^2 \\ +(f+gx^n)u\partial x^2 \\ +[xx(a+bx^n)\partial\partial v+x(c+ex^n)\partial x\partial v+(f+gx^n)v\partial x^2]lkx \end{array}\right\}=0,$$

ubi partem postremam logarithmo affectam seorsim nihilo aequari oportet. Quare posito

$$v=\mathrm{A}x^\lambda+\mathrm{B}x^{\lambda+n}+\mathrm{C}x^{\lambda+2n}+\mathrm{D}x^{\lambda+3n}+\text{etc.}$$

exponenti λ ex aequatione

$$\lambda(\lambda-1)a+\lambda c+f=0,$$

is valor tribuatur, qui nulli incommodo est obnoxius; eritque pro reliquis coëfficientibus,

$$\text{ponendo } \lambda(\lambda-1)b+\lambda e+g=nh,$$

ut sequitur

$$[(n+2\lambda-1)a+c]\mathrm{B}+h\mathrm{A}=0,$$

$$2[(2n+2\lambda-1)a+c]\mathrm{C}+[(n+2\lambda-1)b+e]\mathrm{B}+h\mathrm{B}=0,$$

$$3[(3n+2\lambda-1)a+c]\mathrm{D}+2[(2n+2\lambda-1)b+e]\mathrm{C}+h\mathrm{C}=0,$$

$$4[(4n+2\lambda-1)a+c]\mathrm{E}+3[(3n+2\lambda-1)b+e]\mathrm{D}+h\mathrm{D}=0,$$

etc.

His coëfficientibus ita definitis, quorum primus A arbitrio nostro relinquitur, ponamus

$$u=\Delta+\mathfrak{A}x^\lambda+\mathfrak{B}x^{\lambda+n}+\mathfrak{C}x^{\lambda+2n}+\mathfrak{D}x^{\lambda+3n}+\text{etc.}$$

qui valor si in priori aequatione cum serie pro v inventa substituatur, sequentes series ad nihilum reduci oportet

$$\begin{array}{llll}
xx(a+bx^n)\frac{\partial\partial\Delta}{\partial x^2} & +\lambda(\lambda-1)\mathfrak{A}ax^\lambda & +(\lambda+n)(\lambda+n-1)\mathfrak{B}ax^{\lambda+n} & +(\lambda+2n)(\lambda+2n-1)\mathfrak{C}ax^{\lambda+2n}+\text{etc.} \\
+x(c+ex^n)\frac{\partial\Delta}{\partial x} & & +\lambda(\lambda-1)\mathfrak{A}b & +(\lambda+n)(\lambda+n-1)\mathfrak{B}b \\
 & +\lambda\mathfrak{A}c & +(\lambda+n)\mathfrak{B}c & +(\lambda+2n)\mathfrak{C}c \\
+(f+gx^n)\Delta & & +\lambda\mathfrak{A}e & +(\lambda+n)\mathfrak{B}e \\
 & +\mathfrak{A}f & +\mathfrak{B}f & +\mathfrak{C}f \\
 & & +\mathfrak{A}g & +\mathfrak{B}g \\
 & +2\lambda\mathrm{A}a & +2(\lambda+n)\mathrm{B}a & +2(\lambda+2n)\mathrm{C}a \\
 & & +2\lambda\mathrm{A}b & +2(\lambda+n)\mathrm{B}b \\
 & +\mathrm{A}(c-a) & +\mathrm{B}(c-a) & +\mathrm{C}(c-a) \\
 & & +\mathrm{A}(e-b) & +\mathrm{B}(e-b)
\end{array}$$

Cum autem sit

$$\lambda(\lambda-1)a+\lambda c+f=0 \text{ et } \lambda(\lambda-1)b+\lambda e+g=nh,$$

expressio haec transmutabitur in hanc formam

$$xx(a+bx^n)\frac{\partial\partial\Delta}{\partial x^2}+x(c+ex^n)\frac{\partial\Delta}{\partial x}+(f+gx^n)\Delta$$

$$\begin{array}{lll}
[(2\lambda-1)a+c]\mathrm{A}x^\lambda & +[(2n+2\lambda-1)a+c]\mathrm{B}x^{\lambda+n} & +[(4n+2\lambda-1)a+c]\mathrm{C}x^{\lambda+2n}+\text{etc} \\
 & +[(2\lambda-1)b+e]\mathrm{A} & +[(2n+2\lambda-1)b+e]\mathrm{B} \\
 & +n[(n+2\lambda-1)a+c]\mathfrak{B} & +2n[(2n+2\lambda-1)a+c]\mathfrak{C} \\
 & & +n[(n+2\lambda-1)b+e]\mathfrak{B} \\
 & +nh\mathfrak{A} & +nh\mathfrak{B}
\end{array}$$

ubi Δ denotat quosdam terminos seriei

$$\mathfrak{A}x^\lambda+\mathfrak{B}x^{\lambda+n}+\text{etc.}$$

praemittendos, ita ut ordine retrogrado sit

$$\Delta=\mathfrak{a}x^{\lambda-n}+\mathfrak{b}x^{\lambda-2n}+\mathfrak{c}x^{\lambda-3n}+\ldots+\mathfrak{j}x^{\lambda-in}.$$

Quod principium quomodo quovis casu sit constituendum, sequentia sunt observanda.

I. Principium hoc locum habere nequit, nisi fuerit

$$(\lambda - in)(\lambda - in - 1)a + (\lambda - in)c + f = 0,$$

cum igitur sit

$$\lambda(\lambda - 1)a + \lambda c + f = 0,$$

inde erit

$$\lambda = in + \frac{a - c - \sqrt{[(a-c)^2 - 4af]}}{2a},$$

hinc vero

$$\lambda = \frac{a - c + \sqrt{[(a-c)^2 - 4af]}}{2a},$$

quandoquidem hi duo valores convenire nequeunt, nisi ibi signum radicale negative, hic vero positive accipiatur. Aequatis autem his valoribus fit

$$in = \frac{1}{a}\sqrt{[(a-c)^2 - 4af]}, \text{ seu}$$

$$iinnaa = (a-c)^2 - 4af, \text{ hincque}$$

$$f = \frac{(a-c)^2}{4a} - \tfrac{1}{4}iinna,$$

unde fit

$$\lambda = \frac{a-c}{2a} + \tfrac{1}{2}in$$

Quare si aequatio proposita ita fuerit comparata, ut sit

$$ina = \sqrt{[(a-c)^2 - 4af]},$$

tum sumto

$$\lambda = \frac{a-c}{2a} + \tfrac{1}{2}in,$$

ac pro v sumta serie

$$v = Ax^\lambda + Bx^{\lambda+n} + \text{etc.}$$

alteram seriem u ita constitui convenit

$$u = \mathfrak{j}x^{\lambda - in} + \ldots\ldots + \mathfrak{a}x^{\lambda - n} + \mathfrak{A}x^\lambda + \mathfrak{B}x^{\lambda+n}$$
$$+ \mathfrak{C}x^{\lambda+2n} + \text{etc.}$$

Hic est casus, quo bini valores ipsius λ ex aequatione

$$\lambda(\lambda-1)a+\lambda c+f=0,$$

differentiam habent per n divisibilem, ubi notandum, seriem v a majore valore ipsius λ, seriem vero u a minore inchoari debere

II. Principium Δ omitti nequit, nisi fuerit

$$(2\lambda-1)a+c=0,$$

quo casu fit $\lambda=\frac{a-c}{2}$: atque hic est casus, quo aequationis

$$\lambda(\lambda-1)a+\lambda c+f=0$$

binae radices fiunt inter se aequales, ideoque $f=\frac{(a-c)^2}{4a}$. Continetur ergo hic casus in praecedente, sumendo ibi $i=0$. Quare hoc modo resolventur casus, quibus bini valores ipsius λ vel sunt inter se aequales, vel differentiam habent per exponentem n divisibilem. Sicque reperitur integrale completum per duas series ascendentes v et u expressum, quarum illa v per lx multiplicatur.

Corollarium 1.

974. Quando ergo in aequatione proposita coëfficientes a, c et f ita sunt comparati, ut aequationis

$$\lambda(\lambda-1)a+\lambda c+f=0$$

radices sint $\lambda=\mu$ et $\lambda=\mu-in$, denotante i numerum integrum positivum, integrale completum hujusmodi habebit formam $y=u+\alpha v+vlx$.

Corollarium 2.

975. Hic autem binae quantitates v et u ex his aequationibus

$$\text{I.}\quad xx(a+bx^n)\partial\partial v + x(c+ex^n)\partial x\partial v + (f+gx^n)v\partial x^2 = 0,$$

$$\text{II.}\quad \left.\begin{array}{r} xx(a+bx^n)\partial\partial u + x(c+ex^n)\partial x\partial u + (f+gx^n)u\partial x^2 \\ +2x(a+bx^n)\partial x\partial v - (a+bx^n)v\partial x^2 \\ +(c+ex^n)v\partial x^2 \end{array}\right\} = 0,$$

ita determinari poterunt, ut ponatur

$$v = \mathrm{A}x^{\mu} + \mathrm{B}x^{\mu+n} + \mathrm{C}x^{\mu+2n} + \mathrm{D}x^{\mu+3n} + \text{etc.}$$

$$u = \mathfrak{A}x^{\mu-in} + \mathfrak{B}x^{\mu-in+n} + \mathfrak{C}x^{\mu-in+2n} + \mathfrak{D}x^{\mu-in+3n} + \text{etc.}$$

Has scilicet series substituendo omnes coëfficientes ex uno definire licebit.

Scholion.

976. Logarithmo ergo ipsius x in subsidium vocato, his casibus, quos commemoravimus, integrale completum aequationis propositae per series ascendentes exhiberi potest, dum sine hoc artificio integrale tantum particulare invenitur. Quando enim aequatio $\lambda(\lambda-1)a + \lambda c + f = 0$ duas radices habet, quarum differentia per exponentem n est divisibiles, puta $\lambda = \mu$ et $\lambda = \mu - in$, priore methodo sola series, quae incipit a potestate x^{μ}, determinari potest; si enim altera a potestate $x^{\mu-in}$ incipiens pro y assumeretur, coëfficiens cujusdam termini reperiretur infinitus, unde sequentes omnes forent quoque infiniti, quod incommodum introducendo logarithmus ipsius x feliciter tollitur. Hunc igitur usum istius resolutionis aliquot exemplis illustrasse juvabit.

Exemplum 1.

977. *Aequationis differentio-differentialis*

$$x\partial\partial y + \partial x\partial y + gx^{n-1}y\partial x^2 = 0$$

integrale completum per series ascendentes exhibere.

Hanc aequationem ad nostram formam reducendo habebimus

$$xx\partial\partial y + x\partial x\partial y + gx^n y\partial x^2 = 0,$$

ubi ergo est $a = 1$, $b = 0$, $c = 1$, $e = 0$ et $f = 0$. Hinc $\lambda(\lambda - 1) + \lambda = 0$, seu $\lambda\lambda = 0$, ita ut bini valores ipsius λ sint aequales et $= 0$. Quare posito $y = u + \alpha v + v l x$, resolvi oportet has aequationes

$$\text{I. } xx\partial\partial v + x\partial x\partial v + gx^n v\partial x^2 = 0 \text{ et}$$

$$\text{II. } \left.\begin{array}{l} xx\partial\partial u + x\partial x\partial u + gx^n u\partial x^2 \\ \quad + 2x\partial x\partial v. \end{array}\right\} = 0$$

Statuamus ergo

$$v = A + Bx^n + Cx^{2n} + Dx^{3n} + \text{etc. et}$$

$$u = \mathfrak{A} + \mathfrak{B}x^n + \mathfrak{C}x^{2n} + \mathfrak{D}x^{3n} + \text{etc.}$$

ac prior aequatio praebet

$$\left.\begin{array}{llll} n(n-1)Bx^n & + 2n(2n-1)Cx^{2n} & + 3n(3n-1)Dx^{3n} & + \text{etc.} \\ + nB & + 2nC & + 3nD & \\ + Ag & + Bg & + Cg & \end{array}\right\} = 0,$$

unde fit

$$B = \frac{-Ag}{nn},\ C = \frac{-Bg}{4nn},\ D = \frac{-Cg}{9nn},\ E = \frac{-Dg}{16\alpha n},\ \text{etc.}$$

Tum vero altera aequatio dat

$$\left.\begin{array}{llll} nn\mathfrak{B}x^n & + 4nn\mathfrak{C}x^{2n} & + 9nn\mathfrak{D}x^{3n} & + \text{etc.} \\ + \mathfrak{A}g & + \mathfrak{B}g & + \mathfrak{C}g & \\ + 2nB & + 4nC & + 6nD & \end{array}\right\} = 0,$$

unde colligitur

$$\mathfrak{B} = \frac{-\mathfrak{A}g}{nn} - \frac{2B}{n},\ \mathfrak{C} = \frac{-\mathfrak{B}g}{4nn} - \frac{2C}{2n},\ \mathfrak{D} = \frac{-\mathfrak{C}g}{9nn} - \frac{2D}{3n},\ \text{etc.}$$

Hic autem tuto assumere licet $\mathfrak{A} = 0$, quoniam termini ex $\mathfrak{A}$ oriundi, continentur in membro αv. Cum igitur sit

$$B = \frac{-Ag}{n^2},\ C = \frac{+Ag^2}{1\cdot 4n^4},\ D = \frac{-Ag^3}{1\cdot 4\cdot 9n^6},\ E = \frac{+Ag^4}{1\cdot 4\cdot 9\cdot 16n^8},\ \text{etc.}$$

erit ut sequitur

$$\mathfrak{B} = \frac{2Ag}{n^3}, \quad \mathfrak{C} = \frac{-2Agg}{4n^5} - \frac{2Agg}{2.1.4n^5} = \frac{-6Agg}{2.1.4n^5},$$

$$\mathfrak{D} = \frac{6Ag^3}{2.1.4.9n^7} + \frac{2Ag^3}{3.1.4.9n^7} = \frac{22Ag^3}{2.3.1.4.9n^7},$$

$$\mathfrak{E} = \frac{-22Ag^4}{2.3.1.4.9.16n^9} - \frac{2Ag^4}{4.1.4.9.16n^9} = \frac{-100Ag^4}{2.3.4.1.4.9.16n^9},$$

$$\mathfrak{F} = \frac{100Ag^5}{2.3.4.1.4.9.16.25n^{11}} + \frac{2Ag^5}{5.1.4.9.16.25n^{11}} = \frac{548Ag^5}{2.3.4.5.1.4.9.16.25n^{11}}$$

sicque obtinentur sequentes valores

$$\mathfrak{B} = \frac{2Ag}{n^3}, \quad \mathfrak{C} = \frac{-6Ag^2}{1.8n^5}, \quad \mathfrak{D} = \frac{22Ag^3}{1.8.27n^7}, \quad \mathfrak{E} = \frac{-100Ag^4}{1.8.27.64n^9},$$

$$\mathfrak{F} = \frac{548Ag^5}{1.8.27.64.125n^{11}}, \quad \mathfrak{G} = \frac{-3528Ag^6}{1.8.27.64.125.216n^{13}}, \text{ etc.}$$

ubi numeratores 2, 6, 22, 100, 548, 3528, etc. singuli ita per binos praecedentes definiuntur

$$6 = 3.2 - 1.0, \quad 22 = 5.6 - 4.2, \quad 100 = 7.22 - 9.6,$$

$$548 = 9.100 - 16.22, \quad 3528 = 11.548 - 25.100 \text{ etc.}$$

Consequenter integrale ita exprimetur

$$y = \frac{2Ag}{n^3}x^n - \frac{6Ag^2}{1.8n^5}x^{2n} + \frac{22Ag^3}{1.8.27n^7}x^{3n} - \frac{100Ag^4}{1.8.27.64n^9}x^{4n} + \text{etc.}$$

$$+ A\left(1 - \frac{g}{nn}x^n + \frac{g^2}{1.4n^4}x^{2n} - \frac{g^3}{1.4.9n^6}x^{3n} + \frac{g^4}{1.4.9.16n^8}x^{4n} - \text{etc.}\right)lx$$

$$+ \alpha - \frac{\alpha g}{nn}x^n + \frac{\alpha g^2}{1.4n^4}x^{2n} - \frac{\alpha g^3}{1.4.9n^6}x^{3n} + \frac{\alpha g^4}{1.4.9.16n^8}x^{4n} - \text{etc.}$$

ubi A et α sunt binae constantes arbitrariae.

Exemplum 2.

978. *Aequationis differentio-differentialis*

$$x(1 - xx)\partial\partial y - (1 + xx)\partial x\partial y + xy\partial x^2 = 0,$$

integrale completum per series ascendentes assignare.

Ad formam nostram reducta est

$$xx(1 - xx)\partial\partial y - x(1 + xx)\partial x\partial y + xxy\partial x^2 = 0,$$

ita ut sit $n = 2$, $a = 1$, $b = -1$, $c = -1$, $e = -1$, $f = 0$ et $g = 1$, unde aequationis $\lambda(\lambda - 1) - \lambda = 0$ radices sint $\lambda = 0$ et $\lambda = 2$, quarum differentia per $n = 2$ divisa dat 1.

Posito ergo $y = u + \alpha v + v\,l\,x$, statui debet

$$v = A x^2 + B x^4 + C x^6 + D x^8 + \text{etc. et}$$
$$u = \mathfrak{A} + \mathfrak{B} x^2 + \mathfrak{C} x^4 + \mathfrak{D} x^6 + \mathfrak{E} x^8 + \text{etc.}$$

quae series ex sequentibus aequationibus determinari debent

$$\text{I. } x x (1 - x x) \partial \partial v - x (1 + x x) \partial x \partial v + x x v \partial x^2 = 0,$$

$$\text{II. } \left.\begin{array}{r} x x (1 - x x) \partial \partial u - x (1 + x x) \partial x \partial u + x x u \partial x^2 \\ + 2 x (1 - x x) \partial x \partial v - 2 v \partial x^2 \end{array}\right\} = 0.$$

Hinc pro prioris determinatione fit

$$\left.\begin{array}{rrrrr} & 2 A x^2 & + 12 B x^4 & + 30 C x^6 & + 56 D x^8 + \text{etc.} \\ & & - 2 A & - 12 B & - 30 C \\ - 2 A & & - 4 B & - 6 C & - 8 D \\ & & - 2 A & - 4 B & - 6 C \\ & & + A & + B & + C \end{array}\right\} = 0,$$

ideoque

$$2.4 B = 1.3 A,\quad 4.6 C = 3.5 B,\quad 6.8 D = 5.7 C, \text{ etc.}$$

seu

$$C = \frac{1.3}{2.4} A,\quad C = \frac{1.3.3.5}{2.4.4.6} A,\quad D = \frac{1.3.3.5.5.7}{2.4.4.6.6.8} A, \text{ etc.}$$

Ex altera vero aequatione reperitur

$$\left.\begin{array}{rrrrr} & 2 \mathfrak{B} x^2 & + 12 \mathfrak{C} x^4 & + 30 \mathfrak{D} x^6 & + 56 \mathfrak{E} x^8 + \text{etc.} \\ & - 2 \mathfrak{B} & - 12 \mathfrak{C} & - 30 \mathfrak{D} & \\ - 2 \mathfrak{B} & - 4 \mathfrak{C} & - 6 \mathfrak{D} & - 8 \mathfrak{E} & \\ & - 2 \mathfrak{B} & - 4 \mathfrak{C} & - 6 \mathfrak{D} & \\ + \mathfrak{A} & + \mathfrak{B} & + \mathfrak{C} & + \mathfrak{D} & \\ + 4 A & + 8 B & + 12 C & + 16 D & \\ & - 4 A & - 8 B & - 12 C & \\ - 2 A & - 2 B & - 2 C & - 2 D & \end{array}\right\} = 0$$

unde fieri oportet . . . .

$\mathfrak{A} + 2A = 0, \quad 2.4\,\mathfrak{C} - 1.3\,\mathfrak{B} + 6B - 4A = 0,$
$4.6\,\mathfrak{D} - 3.5\,\mathfrak{C} + 10C - 8B = 0,$
$6.8\,\mathfrak{E} - 5.7\,\mathfrak{D} + 14D - 12C = 0$, etc.

seu cum sit

$$B = \tfrac{1.3}{2.4}A, \quad C = \tfrac{3.5}{4.6}B, \quad D = \tfrac{5.7}{6.8}C, \text{ etc.}$$

erit $\mathfrak{A} = -2A$, tum vero

$$2.\,4\,\mathfrak{C} - 1.\,3\,\mathfrak{B} - \tfrac{2.7}{2.4}A = 0, \quad \mathfrak{C} = \tfrac{1.3}{2.4}\mathfrak{B} + \tfrac{2.7}{2^2.4^2}A,$$
$$4.\,6\,\mathfrak{D} - 3.\,5\,\mathfrak{C} - \tfrac{2.21}{4.6}B = 0, \quad \mathfrak{D} = \tfrac{3.5}{4.6}\mathfrak{C} + \tfrac{2.21}{4^2.6^2}B,$$
$$6.\,8\,\mathfrak{E} - 5.\,7\,\mathfrak{D} - \tfrac{2.43}{6.8}C = 0, \quad \mathfrak{E} = \tfrac{5.7}{6.8}\mathfrak{D} + \tfrac{2.43}{6^2.8^2}C,$$
$$8.\,10\,\mathfrak{F} - 7.\,9\,\mathfrak{E} - \tfrac{2.73}{8.10}D = 0, \quad \mathfrak{F} = \tfrac{7.9}{8.10}\mathfrak{E} + \tfrac{2.73}{8^2.10^2}D.$$

Dum ergo capiatur $\mathfrak{A} = -2A$, littera $\mathfrak{B}$ pro lubitu accipi potest; nihilque impedit, quo minus nihilo aequalis statuatur, siquidem constans α supra est inducta.

Exemplum 3.

979. *Aequationis differentio-differentialis*

$$xx(1+bxx)\partial\partial y + x(-5+exx)\partial x\partial y + (5+gxx)y\partial x^2 = 0$$

integrale completum per series ascendentes exhibere.

Quia hic est $a=1$, $c=-5$ et $f=5$, aequatio $\lambda(\lambda-1)-5\lambda+5=0$, seu $\lambda\lambda-6\lambda+5=0$, radices habet $\lambda=1$ et $\lambda=5$, quarum differentia 4 per $n=2$ dividi potest. Posito ergo $y=u+\alpha v+vlx$ statuatur

$$v = Ax^5 + Bx^7 + Cx^9 + Dx^{11} + Ex^{13} + \text{etc.}$$
$$u = \mathfrak{A}x + \mathfrak{B}x^3 + \mathfrak{C}x^5 + \mathfrak{D}x^7 + \text{etc. et}$$

aequationes resolvendae erunt

I. $xx(1+bxx)\partial\partial v + x(-5+exx)\partial x\partial v + (5+gxx)v\partial x^2 = 0$ et

II. $$\left.\begin{array}{l} xx(1+bxx)\partial\partial u + x(-5+exx)\partial x\partial u + (5+gxx)u\partial x^2 \\ + 2x(1+bxx)\partial x\partial v - (1+bxx)v\partial x^2 \\ + (-5+exx)v\partial x^2 \end{array}\right\} = 0.$$

ubi prior ducit ad

$$\left.\begin{array}{rrrrr}
5.4\,Ax^5 & +7.6\,Bx^7 & +9.8\,Cx^9 & +11.10\,Dx^{11} & +\text{etc.}\\
-5.5\,A & -5.7\,B & -5.9\,C & -5.11\,D & \\
+5\,A & +5\,B & +5\,C & +5\,D & \\
 & +5.4\,Ab & +7.6\,Bb & +9.8\,Cb & \\
 & +5\,Ae & +7\,Be & +9\,Ce & \\
 & +Ag & +Bg & +Cg &
\end{array}\right\}=0,$$

posterior vero ad

$$\left.\begin{array}{rrrrrr}
 & +2.3\mathfrak{B}x^3 & +4.5\mathfrak{C}x^5 & +6.7\mathfrak{D}x^7 & +8.9\mathfrak{E}x^9 & +\text{etc.}\\
-5\mathfrak{A}x & -5.3\mathfrak{B} & -5.5\mathfrak{C} & -5.7\mathfrak{D} & -5.9\mathfrak{E} & \\
+5\mathfrak{A} & +5\mathfrak{B} & +5\mathfrak{C} & +5\mathfrak{D} & +5\mathfrak{E} & \\
 & & +2.3\mathfrak{B}b & +4.5\mathfrak{C}b & +6.7\mathfrak{D}b & \\
 & +\mathfrak{A}e & +3\mathfrak{B}e & +5\mathfrak{C}e & +7\mathfrak{D}e & \\
 & +\mathfrak{A}g & +\mathfrak{B}g & +\mathfrak{C}g & +\mathfrak{D}g & \\
 & & +2.5A & +2.7B & +2.9C & \\
 & & -6A & -6B & -6C & \\
 & & & +2.5Ab & +2.7Bb & \\
 & & & -Ab & -Bb & \\
 & & & +Ae & +Be &
\end{array}\right\}=0$$

Inde fit	seu
$12\,B+A(20\,b+5\,e+g)=0$	$2.\ 6\,B+A(4.5\,b+5e+g)=0,$
$32\,C+B(42\,b+7\,e+g)=0$	$4.\ 8\,C+B(6.7\,b+7e+g)=0,$
$60\,D+C(72\,b+9\,e+g)=0$	$6.\ 10\,D+C(8.9\,b+9e+g)=0,$
etc.	etc.

Hinc autem

$$-4\mathfrak{B}+\mathfrak{A}(e+g)=0,$$

$$0\mathfrak{C}+\mathfrak{B}(2.3b+3e+g)+4A=0,$$

$$2.6\mathfrak{D}+\mathfrak{C}(4.5b+5e+g)+8B+A(9b+e)=0,$$

$$4.8\mathfrak{E}+\mathfrak{D}(6.7b+7e+g)+12C+B(13b+e)=0;$$

etc.

Ex prioribus formulis litterae B, C, D, etc. per A determinantur, ex posteriorum vero secunda fit $\mathfrak{B}=\frac{-4A}{2.3b+3e+g}$, ex prima autem $\mathfrak{A}=\frac{4\mathfrak{B}}{e+g}$, tum vero $\mathfrak{C}$ pro lubitu assumi potest, indeque reliqui coëfficientes $\mathfrak{D}$, $\mathfrak{E}$, $\mathfrak{F}$, etc. definiuntur.

Scholion.

980. Exemplum hoc occasionem nobis suppeditat phaenomena quaedam singularia observandi. Scilicet etiamsi integrale completum in genere lx involvat, tamen id a logarithmo liberum prodit certis casibus. Primo nempe si sit $g=-e$, fit $\mathfrak{B}=0$, manente $\mathfrak{A}$ indefinito, tum vero ob $\mathfrak{B}=0$ capi oportet $A=0$, $B=0$, $C=0$ etc. ideoque $v=0$. Porro vero erit

$$2.\ 6\mathfrak{D}+4\mathfrak{C}(5b+e)=0,$$

$$4.\ 8\mathfrak{E}+6\mathfrak{D}(7b+e)=0,$$

$$6.\ 10\mathfrak{F}+8\mathfrak{E}(9b+e)=0,$$

etc.

ubi $\mathfrak{C}$ altera est constans arbitraria, eritque aequationis

$$xx(1+bxx)\partial\partial y+x(-5+exx)\partial x\partial y+(5-exx)y\partial x^2=0,$$

integrale completum

$$y=\mathfrak{A}x-*+\mathfrak{C}x^5+\mathfrak{D}x^7+\mathfrak{E}x^9+\mathfrak{F}x^{11}+\text{etc.}$$

quod adeo finite exprimitur si $e=-(2i+5)b$, pro i sumendo numeros 0, 1, 2, 3, 4, etc.

Secundo si sit

$$2.\,3b+3e+g=0,\ \text{seu}\ g=-6b-3e,$$

fit $\mathfrak{B}=-\frac{1}{2}\mathfrak{A}(3b+e)$, tum vero $A=0$, $B=0$, $C=0$, etc. ergo $v=0$. Porro vero reperitur

$$\mathfrak{D}=-\tfrac{1}{6}\mathfrak{C}(7b+e),\ \mathfrak{E}=-\tfrac{1}{8}\mathfrak{D}(9b+e),\ \mathfrak{F}=-\tfrac{1}{10}\mathfrak{E}(11b+e)\ \text{etc.}$$

hincque

$$y=\mathfrak{A}x-\tfrac{1}{2}\mathfrak{A}(3b+e)x^3+\mathfrak{C}x^5+\mathfrak{D}x^7+\mathfrak{E}x^9+\text{etc.}$$

ubi $\mathfrak{A}$ et $\mathfrak{C}$ arbitrio nostro reliquuntur.

Tertio si sit

$$4.\,5b+5e+g=0,\ \text{seu}\ g=-20b-5e,$$

primo fit $B=0$, $C=0$, $D=0$, etc. ideoque $v=Ax^5$, tum vero

$\mathfrak{B}=-\mathfrak{A}(5b+e)$, $-\mathfrak{B}(14b+2e)+4A=0$, seu $\mathfrak{B}=\frac{2A}{7b+e}$, hincque $A=-\frac{1}{2}\mathfrak{A}(5b+e)(7b+e)$, porro

$$2.\ 6\mathfrak{D}+A(9b+e)=0,$$
$$4.\ 8\mathfrak{E}+2\mathfrak{D}(11b+e)=0,$$
$$6.\ 10\mathfrak{F}+2\mathfrak{E}(13b+e)=0,$$
etc.

Per $\mathfrak{A}$ ergo definiuntur coëfficientes $\mathfrak{B}$, A, $\mathfrak{D}$, $\mathfrak{E}$, $\mathfrak{F}$, etc. ac $\mathfrak{C}$ quoque arbitrio nostro relinquitur, unde integrale completum hoc casu erit

$$y=Ax^5lx+\mathfrak{C}x^5+\mathfrak{A}x+\mathfrak{B}x^3+*+\mathfrak{D}x^7+\mathfrak{E}x^9+\text{etc.}$$

quae expressio fit finita quoties $(2i+5)b+e=0$.

Exemplum 4.

981. *Si in priori exemplo sit* $e=-7b$ et $g=15b$, *aequationis*

$$xx(1+bxx)\partial\partial y-x(5+7bxx)\partial x\partial y+5(1+3bxx)y\partial x^2=0$$

integrale completum algebraice exhibere.

Erit ergo $\mathfrak{B} = +2\mathfrak{A}b$, $A = 0$, $\mathfrak{D} = 0$, $\mathfrak{E} = 0$, ideoque $v = 0$ et $u = \mathfrak{A}x + 2\mathfrak{A}bx^3 + \mathfrak{C}x^5$, unde pro $\mathfrak{A}$ et $\mathfrak{C}$ sumendo constantes quascunque, erit integrale completum

$$y = \mathfrak{A}x(1 + 2bxx) + \mathfrak{C}x^5.$$

Integralia ergo particularia erunt

$$y = ax(1 + 2bxx),\ y = ax^5,\ y = ax(1 + bxx)^2.$$

Corollarium 1.

982. Posito $y = e^{\int z\partial x}$, ut sit $z = \frac{\partial y}{y\partial x}$, aequationis hujus differentialis primi gradus

$$xx(1 + bxx)\partial z + xx(1 + bxx)zz\partial x - x(5 + 7bxx)z\partial x + 5(1 + 3bxx)\partial x = 0;$$

integrale completum est $z = \frac{\mathfrak{A}(1 + 6bxx) + 5\mathfrak{C}x^4}{\mathfrak{A}x(1 + 2bxx) + \mathfrak{C}x^5}$.

Corollarium 2.

983. Aequatio autem differentio-differentialis integrabilis redditur, si dividatur per $xx(1 + bxx)^2$, eritque integrale

$$\frac{x\partial y - 5y\partial x}{x(1 + bxx)} C\partial x, \text{ seu } \partial y - \frac{5y\partial x}{x} = C\partial x(1 + bxx),$$

quae per x^5 divisa integrale praebet

$$\frac{y}{x^5} = \frac{-C}{4x^4} - \frac{bC}{2x^2} + D \text{ seu}$$
$$y = -\tfrac{1}{4}Cx(1 + 2bxx) + Dx^5,$$

ut ante.

Scholion.

984. Deficit autem adhuc integratio completa nostrae aequationis generalis per series ascendentes, casu quo $a = 0$, ideoque $\lambda c + f = 0$, unde unicus pro exponente λ valor definitur $\lambda = \frac{-f}{c}$, qui tantum integrale particulare suppeditat, atque hoc etiam tollitur, si fuerit $c = 0$. Quia autem his casibus $a = 0$, coëfficiens b

certo adsit necesse est, ex quo integrale completum per series descendentes exhiberi poterit, cum aequatio $\lambda (\lambda - 1) b + \lambda e + g = 0$ duas semper contineat radices, ex quibus duplex series obtinetur. Simile autem hic incommodum usu venire potest, quando binae radices ipsius λ vel prodeunt aequales, vel differentiam habent per exponentem n divisibilem. Verum huic incommodo, seriem per $l x$ multiplicatam introducendo, simili methodo medela affertur, qua in hoc problemate sumus usi, ac superfluum foret istam evolutionem hic repetere. Quodsi autem binae radices ipsius λ tam pro seriebus ascendentibus quam descendentibus fiant imaginariae, ostendendum restat, quomodo integrale completum per series infinitas exprimi oporteat.

Problema 124.

985. Proposita aequatione differentio-differentiali

$$x x (a + b x^n) \partial \partial y + x (c + e x^n) \partial x \partial y + (f + g x^n) y \partial x^2 = 0,$$

si eveniat ut aequatio

$$\lambda (\lambda - 1) a + \lambda c + f = 0$$

radices habeat imaginarias, ejus integrale completum per series ascendentes exhibere.

Solutio.

Ex supra allatis (971) colligitur hoc casu statui debere

$$y = v \sin. \mu l x + u \cos. \mu l x \text{ unde fit}$$

$$\partial y = \left(\partial v - \frac{\mu u \partial x}{x}\right) \sin. \mu l x + \left(\frac{\mu v \partial x}{x} + \partial u\right) \cos. \mu l x, \text{ et}$$

$$\partial \partial y = \left(\partial \partial v - \frac{2 \mu \partial x \partial u}{x} + \frac{\mu u \partial x^2}{x x} - \frac{\mu \mu v \partial x^2}{x x}\right) \sin. \mu l x$$
$$+ \left(\partial \partial u + \frac{2 \mu \partial x \partial v}{x} - \frac{\mu v \partial x^2}{x x} - \frac{\mu \mu u \partial x^2}{x x}\right) \cos. \mu l x,$$

qua facta substitutione, si terminos tam sin. $\mu l x$ quam cos. $\mu l x$ affectos seorsim ad nihilum redigamus, obtinebimus duas sequentes aequationes

$$\text{I.}\quad \left.\begin{aligned} xx(a+bx^n)\partial\partial v + x(c+ex^n)\partial x\partial v + (f+gx^n)v\partial x^2 \\ -2\mu x(a+bx^n)\partial x\partial u - \mu\mu(a+bx^n)v\partial x^2 \\ +\mu(a+bx^n)u\partial x^2 \\ -\mu(c+ex^n)u\partial x^2 \end{aligned}\right\} = 0$$

$$\text{II.}\quad \left.\begin{aligned} xx(a+bx^n)\partial\partial u + x(c+ex^n)\partial x\partial u + (f+gx^n)u\partial x^2 \\ +2\mu x(a+bx^n)\partial x\partial v - \mu\mu(a+bx^n)u\partial x^2 \\ -\mu(a+bx^n)v\partial x^2 \\ +\mu(c+ex^n)v\partial x^2 \end{aligned}\right\} = 0$$

Jam, pro v et u assumamus has series ascendentes

$$v = \mathrm{A}x^\lambda + \mathrm{B}x^{\lambda+n} + \mathrm{C}x^{\lambda+2n} + \mathrm{D}x^{\lambda+3n} + \text{etc.}$$
$$u = \mathfrak{A}x^\lambda + \mathfrak{B}x^{\lambda+n} + \mathfrak{C}x^{\lambda+2n} + \mathfrak{D}x^{\lambda+3n} + \text{etc.}$$

iisque substitutis, prior aequatio abit in hanc

$\lambda(\lambda-1)\mathrm{A}ax^\lambda$	$+(\lambda+n)(\lambda+n-1)\mathrm{B}ax^{\lambda+n}$	$+(\lambda+2n)(\lambda+2n-1)\mathrm{C}ax^{\lambda+2n}$
	$+\lambda(\lambda-1)\mathrm{A}b$	$+(\lambda+n)(\lambda+n-1)\mathrm{B}b$
$+\lambda\mathrm{A}c$	$+(\lambda+n)\mathrm{B}c$	$+(\lambda+2n)\mathrm{C}c$
	$+\lambda\mathrm{A}e$	$+(\lambda+n)\mathrm{B}e$
$+\mathrm{A}f$	$+\mathrm{B}f$	$+\mathrm{C}f$
	$+\mathrm{A}g$	$+\mathrm{B}g$
$-2\mu\lambda\mathfrak{A}a$	$-2\mu(\lambda+n)\mathfrak{B}a$	$-2\mu(\lambda+2n)\mathfrak{C}a$
	$-2\mu\lambda\mathfrak{A}b$	$-2\mu(\lambda+n)\mathfrak{B}b$
$-\mu\mu\mathrm{A}a$	$+\mu\mu\mathrm{B}a$	$-\mu\mu\mathrm{C}a$
	$-\mu\mu\mathrm{A}b$	$-\mu\mu\mathrm{B}b$
$+\mu\mathfrak{A}a$	$+\mu\mathfrak{B}a$	$+\mu\mathfrak{C}a$
	$+\mu\mathfrak{A}b$	$+\mu\mathfrak{B}b$
$-\mu\mathfrak{A}c$	$-\mu\mathfrak{B}c$	$-\mu\mathfrak{C}c$
	$-\mu\mathfrak{A}e$	$-\mu\mathfrak{B}e$

Hinc altera aequatio facile formatur permutandis litteris latinis et germanicis atque insuper signum numeri μ mutando.

Utrinque ergo potestas prima x^{λ} exigit has aequationes

$$A[\lambda(\lambda-1)a+\lambda c+f-\mu\mu a]-\mu\mathfrak{A}(2\lambda a-a+c)=0$$

$$\mathfrak{A}[\lambda(\lambda-1)a+\lambda c+f-\mu\mu a]+\mu A(2\lambda a-a+c)=0,$$

unde necesse est ut sit

$$\text{tam } 2\lambda a-a+c=0$$

$$\text{quam } \lambda(\lambda-1)a+\lambda c+f-\mu\mu a=0.$$

Inde fit $\lambda=\frac{1}{2}-\frac{c}{2a}$; qui valor hic substitutus dat

$$-a\left(\frac{1}{4}-\frac{cc}{4aa}\right)+\frac{c}{2}-\frac{cc}{2a}+f=\mu\mu a=-\frac{a}{4}+\frac{c}{2}-\frac{cc}{4a}+f, \text{ seu}$$

$$\mu\mu a=\frac{4af-(a-c)^2}{4a}, \text{ ideoque}$$

$$\mu=\frac{\sqrt{[4af-(a-c)^2]}}{2a} \text{ et } \lambda=\frac{a-c}{2a}.$$

Unde patet hanc solutionem locum habere si $4af>(a-c)^2$, quo ipso casu praecedens solutio fiebat imaginaria. Hic autem quantitates A et $\mathfrak{A}$ arbitrio nostro relinquuntur.

Terminus vero $x^{\lambda+n}$ utrinque postulat has aequationes

$$B[(\lambda+n)(\lambda+n-1)a+(\lambda+n)c+f-\mu\mu a]+A[\lambda(\lambda-1)b+\lambda e+g-\mu\mu b]$$
$$-\mu\mathfrak{B}[2(\lambda+n)a-a+c]-\mu\mathfrak{A}(2\lambda b-b+e)=0 \text{ et}$$

$$\mathfrak{B}[(\lambda+n)(\lambda+n-1)a+(\lambda+n)c+f-\mu\mu a]+\mathfrak{A}[\lambda(\lambda-1)b+\lambda e+g-\mu\mu b]$$
$$+\mu B[2(\lambda+n)a-a+c]+\mu A(2\lambda b-b+e)=0.$$

Sit brevitatis gratia

$$(\lambda+n)(\lambda+n-1)a+(\lambda+n)c+f-\mu\mu a=nna=\alpha$$

$$\lambda(\lambda-1)b+\lambda e+g-\mu\mu b=\beta$$

$$2(\lambda+n)a-a+c=2na=\gamma$$

$$2\lambda b-b+e=\delta,$$

ut habeamus

$$B\alpha + A\beta - \mu\mathfrak{B}\gamma - \mu\mathfrak{A}\delta = 0 \text{ et}$$
$$\mathfrak{B}\alpha + \mathfrak{A}\beta + \mu B\gamma + \mu A\delta = 0,$$

unde colligitur

$$B = \frac{-A(\alpha\beta + \mu\mu\gamma\delta) + \mu\mathfrak{A}(\alpha\delta - \beta\gamma)}{\alpha\alpha + \mu\mu\gamma\gamma} \text{ et}$$
$$\mathfrak{B} = \frac{-\mathfrak{A}(\alpha\beta + \mu\mu\gamma\delta) - \mu A(\alpha\delta - \beta\gamma)}{\alpha\alpha + \mu\mu\gamma\gamma}.$$

At vero ex valoribus assumtis est

$$\alpha = nna,\ \beta = \frac{(ae - bc)(a - c)}{aa} - \frac{bf}{a} + g,\ \gamma = 2na,\ \delta = \frac{ae - bc}{a},$$

unde ex assumtis A et $\mathfrak{A}$ definiuntur B et $\mathfrak{B}$, hincque porro C, $\mathfrak{C}$; D, $\mathfrak{D}$ etc.

Exemplum 1.

986. *Sit $c = a$ et $f = a$, ut fiat $\mu = 1$, et investigetur integrale hujus aequationis*

$$xx(a + bx^n)\partial\partial y + x(a + ex^n)\partial x\partial y + (a + gx^n)y\partial x^2 = 0.$$

Hic ergo erit $\lambda = 0$ et $\mu = 1$, unde posito

$$y = v \sin.\ lx + u \cos.\ lx,$$

ac pro v et u sumtis seriebus

$$v = A + Bx^n + Cx^{2n} + Dx^{3n} + \text{etc.}$$
$$u = \mathfrak{A} + \mathfrak{B}x^n + \mathfrak{C}x^{2n} + \mathfrak{D}x^{3n} + \text{etc.}$$

coëfficientes A et $\mathfrak{A}$ pro lubitu accipi possunt. Ex iis primo, ob $\alpha = nna$, $\beta = g - b$, $\gamma = 2na$ et $\delta = e - b$, erit

$$B = \frac{-A[nna(g-b) + 2na(e-b)] + \mathfrak{A}[nna(e-b) - 2na(g-b)]}{n^4aa + 4nnaa} \text{ seu}$$
$$B = \frac{-A[n(g-b) + 2(e-b)] + \mathfrak{A}[n(e-b) - 2(g-b)]}{na(nn+4)} \text{ et}$$
$$\mathfrak{B} = \frac{-\mathfrak{A}[n(g-b) + 2(e-b)] - A[n(e-b) - 2(g-b)]}{na(nn+4)}.$$

Pro sequentibus coëfficientibus habebimus

$$\mathrm{C}\,[2n(2n-1)a+2na+a-a]+\mathrm{B}\,[n(n-1)b+ne+g-b]$$
$$-\mathfrak{C}\,(4na-a+a)-\mathfrak{B}\,(2nb-b+e)=0,\text{ seu}$$

$$4nn\mathrm{C}a+\mathrm{B}[(nn-n-1)b+ne+g]-4n\mathfrak{C}a$$
$$-\mathfrak{B}[(2n-1)b+e]=0,\text{ et}$$

$$4nn\mathfrak{C}a+\mathfrak{B}[(nn-n-1)b+ne+g]+4n\mathrm{C}a$$
$$+\mathrm{B}[(2n-1)b+e]=0,$$

quarum illa per n multiplicata huic addatur, ut prodeat

$$4n(nn+1)\mathrm{C}a+\mathrm{B}[(n^3-nn+n-1)b+(nn+1)e+ng]$$
$$+\mathfrak{B}[-(nn+1)b+g]=0,\text{ hinc}$$

$$\mathrm{C}=\frac{-\mathrm{B}[(n-1)(nn+1)b+(nn+1)e+ng]+\mathfrak{B}[(nn+1)b-g]}{4na(nn+1)}\text{ et}$$

$$\mathfrak{C}=\frac{-\mathfrak{B}[(n-1)(nn+1)b+(nn+1)e+ng]-\mathrm{B}[(nn+1)b-g]}{4na(nn+1)}.$$

Porro erit:

$$9nn\mathrm{D}a+\mathrm{C}[(4nn-2n-1)b+2ne+g]+6n\mathfrak{D}a$$
$$-\mathfrak{C}[(4n-1)b+e]=0$$

$$9nn\mathfrak{D}a+\mathfrak{C}[(4nn-2n-1)b+2ne+g]+6n\mathrm{D}a$$
$$+\mathrm{C}[(4n-1)b+e]=0,$$

quarum illa per $3n$, haec vero per 2 multiplicata junctim dant

$$3n(9nn+4)\mathrm{D}a+\mathrm{C}[(12n^3-6nn+5n-2)b+2(3nn+1)e+3ng]$$
$$+\mathfrak{C}[(-4nn-n-2)b+ne+2g]=0;$$

unde sequitur

$$\mathrm{D}=\frac{-\mathrm{C}[(12n^3-6nn+5n-2)b+2(3nn+1)e+3ng]+\mathfrak{C}[(4nn+n+2)b-ne-2g]}{3n(9nn+4)a}$$

$$\mathfrak{D}=\frac{-\mathfrak{C}[(12n^3-6nn+5n-2)b+2(3nn+1)e+3ng]-\mathrm{C}[(4nn+n+2)b-ne-2g]}{3n(9nn+4)a}.$$

In genere autem ex coëfficientibus quibuscunque M et $\mathfrak{M}$ sequentes N et $\mathfrak{N}$ definiuntur per has formulas

$$in(iinn+4)Na$$
$$+M[[i(i-1)^2n^3-i(i-1)nn+(3i-4)n-2]b+i(i-1)nne+2e+ing]$$
$$-\mathfrak{M}[[2(i-1)nn+(i-2)n+2]b-(i-2)ne-2g]=0$$
$$in(iinn+4)\mathfrak{N}a$$
$$+\mathfrak{M}[[i(i-1)^2n^3-i(i-1)nn+(3i-4)n-2]b+i(i-1)nne+2e+ing]$$
$$+M[[2(i-1)nn+(i-2)n+2]b-(i-2)ne-2g]=0.$$

Corollarium 1.

987. Si quantitates b, e, g ita sint comparatae, ut binae litterae sibi respondentes N et $\mathfrak{N}$ evanescant, sequentes omnes evanescent, et integrale completum forma finita exprimetur. Ita ut B et $\mathfrak{B}$ evanescant, fieri debet

$$2(g-b)=n(e-b) \text{ et } n(g-b)=-2(e-b);$$

unde fit $g=e=b$, et ipsa aequatio proposita factorem habebit $a+bx^n$.

Corollarium 2.

988. In genere autem integrale finite exprimetur, si denotante i numerum integrum quemcunque positivum sit

$$g=[(i-1)nn+\tfrac{1}{2}(i-2)n+1]b-\tfrac{1}{2}(i-2)ne,$$

tum vero

$$e=-[2(i-1)n-1]b,$$

unde fit

$$g=[(i-1)^2nn+1]b.$$

Exemplum 2.

989. *Sumto* $n=1$, *si sit* $e=-b$ et $g=2b$, *hujus aequationis*

$$xx(a+bx)\partial\partial y+x(a-bx)\partial x\partial y+(a+2bx)y\partial x^2=0$$

integrale completum assignare.

Ex formulis modo inventis colligimus

$$B = \frac{-A(g+2e-3b)+\mathfrak{A}(e+b-2g)}{5a} = \frac{3Ab-4\mathfrak{A}b}{5a} \text{ et}$$

$$\mathfrak{B} = \frac{3\mathfrak{A}b+4Ab}{5a};$$

tum vero

$$C = \frac{-B(2e+g)+\mathfrak{B}(2b-g)}{5a} = 0 \text{ et } \mathfrak{C} = 0.$$

Quocirca habebimus

$$v = A + \frac{(3A-4\mathfrak{A})b}{5a}x, \text{ et } u = \mathfrak{A} + \frac{(3\mathfrak{A}+4A)b}{5a}x;$$

hincque integrale completum elicitur

$$y = A\sin.lx + \mathfrak{A}\cos.lx + \frac{bx}{5a}[(3A-4\mathfrak{A})\sin.lx + (3\mathfrak{A}+4A)\cos.lx].$$

Corollarium 1.

990. Sumto $\mathfrak{A} = 0$, habebitur integrale particulare

$$y = A\left(\sin.lx + \frac{3bx}{5a}\sin.lx + \frac{4bx}{5a}\cos.lx\right).$$

Sin autem sit $A = 0$, aliud habebitur

$$y = \mathfrak{A}\left(\cos.lx - \frac{4bx}{5a}\sin.lx + \frac{3bx}{5a}\cos.lx\right).$$

Corollarium 2.

991. Posito $y = e^{\int s\partial x}$, aequatio nostra reducitur ad hanc

$$xx(a+bx)\partial s + xx(a+bx)ss\partial x + x(a-bx)s\partial x + (a+2bx)\partial x = 0,$$

cujus integrale habetur $s = \frac{\partial y}{y\partial x}$ inde definiendum, quae aequatio in plures alias formas transfundi potest.

Scholion.

992. Simili modo integratio per series descendentes instituitur, si exponentes singulorum terminorum prodeant imaginarii; quod

seorsim exposuisse ne opus quidem erit. Atque haec sufficiunt, ut pateat, quibusnam cautelis in resolutione aequationum per series infinitas sit utendum. Summus autem usus istarum evolutionum in hoc consistit, ut aequationes differentio-differentiales exhiberi queant, quarum saltem integrale particulare algebraicum assignare liceat, quos casus supra §. 969 indicavimus. Similis porro integratio per series infinitas pari modo extendi potest ad hujusmodi aequationes

$$xx(a + bx^n + \beta x^{2n})\partial\partial y + x(c + ex^n + \varepsilon x^{2n})\partial x \partial y + (f + gx^n + \gamma x^{2n}) y \partial x^2 = 0$$

tum autem seriei quaesitae quilibet terminus per duos praecedentes determinatur, ita ut si bini contigui evanescant, sequentes omnes in nihilum sint abituri. Quodsi autem terminus ab y vacuus affuerit, resolutio in series fit facilior, cui propterea non immorandum censeo. Veluti si proponatur haec aequatio

$$xx\partial\partial y - x\partial x\partial y + ax^n y\partial x^2 = bx^m \partial x^2,$$

series a potestate x^m est inchoanda, ponendo

$$y = Ax^m + Bx^{m+n} + Cx^{m+2n} + Dx^{m+3n} + \text{etc}$$

unde fit

$$\begin{array}{lll} m(m-1)Ax^m + & (m+n)(m+n-1)Bx^{m+n} + & (m+2n)(m+2n-1)Cx^{m+2n} + \text{etc.} \\ -mA & -(m+n)B & -(m+n)C \\ -b & +Aa & +Ba \end{array}$$

hincque

$$A = \frac{b}{m(m-2)},\quad B = \frac{-Aa}{(m+n)(m+n-2)},\quad C = \frac{-Ba}{(m+2n)(m+2n-2)},\ \text{etc.}$$

ubi quidem multa observanda occurrunt, quae per praecepta supra data expedire licet. Imprimis autem in hoc negotio juvat, aequationem propositam ope substitutionis in alias transformasse, quarum

resolutio per series fiat simplicior, quod cum pluribus modis fieri possit, hoc argumentum sequenti capite diligentius pertractare visum est, idque pro forma aequationum

$$L \partial \partial y + M \partial x \partial y + N y \partial x^2 = 0,$$

quandoquidem pro aliis formis hujusmodi transformatio raro locum invenit.

CAPUT IX.

DE TRANSFORMATIONE AEQUATIONUM DIFFERENTIO-DIFFERENTIALIUM.

$$L\partial\partial y + M\partial x\partial y + Ny\partial x^2 = 0.$$

Problema 125.

993.

Aequationem differentio-differentialem

$$L\partial\partial y + M\partial x\partial y + Ny\partial x^2 = 0,$$

in qua L, M, N sunt functiones quaecunque ipsius x, sumto elemento ∂x constante, ope substitutionis $y = e^{\int P\partial x} z$ in aliam formam transmutare.

Solutio.

Cum hinc sit $\frac{\partial y}{y} = P\partial x + \frac{\partial z}{z}$, erit differentiando

$$\frac{\partial\partial y}{y} - \frac{\partial y^2}{y^2} = \partial x\partial P + \frac{\partial\partial z}{z} - \frac{\partial z^2}{zz}, \text{ ergo}$$

$$\frac{\partial\partial y}{y} = \frac{\partial\partial z}{z} + \frac{2P\partial x\partial z}{z} + \partial x\partial P + PP\partial x^2.$$

Quare cum aequatio nostra sit

$$\frac{L\partial\partial y}{y} + \frac{M\partial x\partial y}{y} + N\partial x^2 = 0,$$

erit facta substitutione

$$\frac{L\partial\partial z}{z} + \frac{2LP\partial x\partial z}{z} + L\partial x\partial P + LPP\partial x^2$$
$$+ \frac{M\partial x\partial z}{z} + MP\partial x^2 + N\partial x^2 = 0,$$

seu per x multiplicando

$$L\partial\partial z + (2LP + M)\partial x\partial z$$
$$+ z\partial x(L\partial P + LPP\partial x + MP\partial x + N\partial x) = 0,$$

ubi pro P functionem quamcunque ipsius x accipere licet, unde innumerabiles aequationes inter binas variabiles x et z obtinentur.

Corollarium 1.

994. Quodsi ergo hanc aequationem transformatam integrare vel per seriem resolvere liceat, ex invento valore ipsius z habebitur $y = e^{\int P\partial x} z$.

Corollarium 2.

995. Aequatio transformata similis est propositae, propterea quod in ea variabilis z cum suis differentialibus ∂z et $\partial\partial z$ ubique unicam dimensionem occupat, perinde ac y in aequatione proposita.

Corollarium 3.

996. Si eveniat, ut ambae aequationes, proposita ac transformata, aeque commode per series resolvi possint, hoc modo plures resolutiones ejusdem aequationis exhiberi possunt.

Scholion 1.

997. Cum aequationes commode per series resolubiles in hac forma contineantur

$$xx(a+bx^n)\partial\partial x + x(c+ex^n)\partial x\partial y + (f+gx^n)y\partial x^2 = 0,$$

ubi est

$$L = xx(a+bx^n),\ M = x(c+ex^n),\ N = f+gx^n,$$

ut transformata similem obtineat formam, fieri oportet $LP = x(\mu + \nu x^n)$, ideoque $P = \frac{\mu + \nu x^n}{x(a+bx^n)}$. Hinc erit

$$\partial P = \frac{-\mu a - \mu (n+1) b x^n + \nu (n-1) a x^n - \nu b x^{2n}}{x x (a + b x^n)^2} \partial x$$

ideoque

$$L \partial P + L P P \partial x + M P \partial x =$$

$$\left\{\begin{array}{ll} -\mu a - (n+1) \mu b x^n + (n-1) \nu a x^n - \nu b x^{2n} & \\ +\mu\mu + 2 \mu \nu x^n & + \nu \nu x^{2n} \\ +\mu c + \mu e x^n + \nu c x^n & + \nu e x^{2n} \end{array}\right\} : a + b x^n$$

ubi divisio per $a + b x^n$ succedere debet. Statuatur quotus $= \mu h + \nu k x^n$; fietque

$$\mu = a - c + a h, \quad \nu = b - e + a k,$$

ac praeterea

$$2 \mu \nu - (n+1) \mu b + (n-1) \nu a + \mu e + \nu c = \mu b h + \nu a k,$$

ubi priores valores substituti praebent

$$(h - k + n)(b c - a e) = n a b (h - k) + a b (h - k)^2;$$

unde fit vel

$$h - k = \frac{b e + a e}{a b}, \text{ vel } h - k = -n.$$

Litterarum ergo b et k altera arbitrio nostro relinquitur, fitque aequatio transformata

$$x x (a + b x^n) \partial \partial z + x [2 \mu + c + (2 \nu + e) x^n] \partial x \partial z$$
$$+ [f + \mu h + (g + \nu k) x^n] z \partial x^2 = 0.$$

Hujus autem resolutio tam per series ascendentes, quam descendentes similes ipsius x postulat potestates: Substitutio autem ipsa fit

$$y = x^{\frac{a - c}{a} + h} (a + b x^n)^{\frac{b c - a e}{n a b} - \frac{(h - k)}{n}} z,$$

ubi ne sola potestas ipsius x ingrediatur, sumi debet $h - k = -n$. Nihil interest quomodo hic h accipiatur, sumto ergo $h = 0$,

fit $k = n$, et substitutio

$$y = x^{\frac{a-c}{a}} (a + b x^n)^{\frac{bc - ae}{nab} + 1} z,$$

quae ducit ad hanc aequationem

$$xx(a+bx^n)\partial\partial z + x[2a - c + (2(n+1)b - e)x^n]\partial x \partial z$$
$$+ [f + (n(n+1)b - ne + g)x^n] z \partial x^2 = 0.$$

Scholion 2.

998. Supra §. 970. vidimus, aequationem propositam inter x et y algebraicum admittere integrale, si fuerit

$$\frac{c}{2a} - \frac{e}{2b} \ldots \frac{\sqrt{[(a-c)^2 - 4af]}}{2a} \ldots \frac{\sqrt{[(b-e)^2 - 4bg]}}{2b} = in,$$

quae si transformata simili modo tractetur, integrale algebraicum assignari poterit, si fuerit

$$-\frac{c}{2a} + \frac{e}{2b} \ldots \frac{\sqrt{[(a-c)^2 - 4af]}}{2a} \ldots \frac{\sqrt{[(b-e)^2 - 4bg]}}{2b} - n = in,$$

quibus conditionibus conjunctis concludere licet, integrale algebraicum satisfacere, dummodo haec formula

$$\frac{bc - aa}{2ab} \ldots \frac{\sqrt{[(a-c)^2 - 4af]}}{2a} \ldots \frac{\sqrt{[(b-e)^2 - 4bg]}}{2b}:$$

divisibilis extiterit per exponentem n, hic signum .. ad ambiguitatem positivi ac negativi designandam adhibui. Quare si ponamus

$$f = \frac{(a-c)^2 - bb}{4a} \text{ et } g = \frac{(b-e)^2 - kk}{4b},$$

integrabilitas locum habet, quoties haec expressio $\frac{bc - ae \pm bh \pm ak}{2nab}$ fuerit numerus integer, sive positivus sive negativus.

Exemplum.

999. *Proposita aequatione*

$$xx(1 - xx)\partial\partial y + x(1 + 2mxx)\partial x \partial y$$
$$- m(m+1)xxy\partial x^2 = 0,$$

invenire casus, quibus integrale algebraicum saltem particulare assignari potest.

Hic est $a = 1$, $b = -1$, $c = 1$, $e = 2m$, $f = 0$, $g = -m(m+1)$ et $n = 2$. Hinc deducimus

$$h = \sqrt{[(a-c)^2 - 4af]} = 0, \text{ et}$$
$$k = \sqrt{[(b-e)^2 - 4bg]} = \sqrt{[(2m+1)^2 - 4m(m+1)]}$$

hoc est $k = \pm 1$. Formula ergo numero integro aequalis est, $\frac{-1-2m\pm 1}{-4}$, unde geminos pro m casus nanciscimur

$$\text{vel } 2m + 2 = \pm 4i, \text{ vel } 2m = \pm 4i, \text{ hoc est}$$
$$\text{vel } m = \pm 2i - 1, \text{ vel } m = \pm 2i,$$

dummodo ergo m sit numerus integer sive positivus sive negativus, integrale particulare algebraicum exhiberi potest. Substitutio autem aequationem transformatam praebens est

$$y = (1 - xx)^{\frac{-1-2m}{-2} + 1} z = (1 - xx)^{\frac{2m+3}{2}} z,$$

ipsa vero aequatio transformata

$$xx(1 - xx)\partial\partial z + x[1 - 2(m+3)xx]\partial x \partial z$$
$$-(m+2)(m+3)xxz\partial x^2 = 0,$$

quam ex illa oriri manifestum est, si loco m scribatur $-m-3$. Ipsa autem haec integralia reperiuntur, ob $\lambda\lambda = 0$, ponendo

$$y = A + Bx^2 + Cx^4 + Dx^6 + Ex^8 + \text{etc.}$$

unde fit

$$\left.\begin{array}{lllll} 2Bxx + 12Cx^4 & + 30Dx^6 & + 56Ex^8 & + \text{etc.} \\ \quad\;\; - 2B & - 12C & - 30D & \\ + 2B + 4C & + 6D & + 8E & \\ \quad\;\; + 4mB & + 8mC & + 12mD & \\ -m(m+1)A - m(m+1)B & - m(m+1)C & - m(m+1)D & \end{array}\right\} = 0.$$

Ergo determinatio coëfficientium ita se habet

$$B = \frac{m(m+1)}{4} A,\ C = \frac{(m-1)(m-2)}{16} B,\ D = \frac{(m-3)(m-4)}{36} C, \text{ etc.}$$

Ac si ponatur

$$z = \mathfrak{A} + \mathfrak{B} x^2 + \mathfrak{C} x^4 + \mathfrak{D} x^6 + \mathfrak{E} x^8 + \text{etc. erit}$$

$$\mathfrak{B} = \frac{(m+2)(m+3)}{4} \mathfrak{A},\ \mathfrak{C} = \frac{(m+4)(m+5)}{16} \mathfrak{B},\ \mathfrak{D} = \frac{(m+6)(m+7)}{36} \mathfrak{C}, \text{ etc.}$$

Problema 126.

1000. Aequationem differentio-differentialem

$$L \partial \partial y + M \partial x \partial y + N y \partial x^2 = 0,$$

ope substitutionis $\frac{\partial y}{y} = \frac{P z \partial x^2}{\partial z}$, in aliam ejusdem formae transmutare.

Solutio.

Hic scilicet quaeritur, qualem functionem ipsius x pro P accipi oporteat, ut facta substitutione variabilis z cum suis differentialibus ∂z et $\partial \partial z$ ubique unicam dimensionem obtineat. Cum igitur sit $\frac{\partial y}{y} = \frac{P z \partial x^2}{\partial z}$ erit differentiando

$$\frac{\partial \partial y}{y} - \frac{\partial y^2}{y^2} = \frac{-P z \partial x^2 \partial \partial z}{\partial y^2} + \frac{z \partial x^2 \partial P}{\partial z} + P \partial x^2 \text{ et}$$

$$\frac{\partial \partial y}{y} = \frac{-P z \partial x^2 \partial \partial z}{\partial z^2} + \frac{z \partial x^2 \partial P}{\partial z} + \frac{P P z z \partial x^4}{\partial z^2} + P \partial x^2,$$

quibus valoribus substitutis fit

$$\frac{L P z \partial x^2 \partial \partial z}{\partial z^2} + \frac{L z \partial x^2 \partial P}{\partial z} + L P \partial x^2 + \frac{L P P z z \partial x^4}{\partial z^2}$$
$$+ \frac{M P z \partial x^3}{\partial z} + N \partial x^2 = 0.$$

Sumamus ergo $L P + N = 0$, seu $P = \frac{-N}{L}$, et multiplicando per $\frac{-\partial z^2}{P z \partial x^2}$, nanciscemur

$$L \partial \partial z - \frac{L \partial P \partial z}{P} - L P z \partial x^2 - M \partial x \partial z = 0, \text{ seu}$$

$$L \partial \partial z - M \partial x \partial z - \frac{L \partial N \partial z}{N} + \partial L \partial z + N z \partial x^2 = 0.$$

Aequatio ergo proposita ope substitutionis $\frac{\partial y}{y} = \frac{-N z \partial x^2}{L \partial x}$ transformatur in hanc

$$L\partial\partial z + \left(\frac{\partial L}{\partial x} - M - \frac{L\partial N}{N\partial x}\right)\partial x\partial z + Nz\partial x^2 = 0.$$

Quodsi ergo hinc valor ipsius z erui possit, habebitur etiam valor ipsius y per x expressus.

Corollarium 1.

1001. Si in hac aequatione transformata vicissim ponatur $\frac{\partial z}{z} = \frac{-Nz\partial x^2}{L\partial y}$, ipsa aequatio proposita exoritur, unde hae duae aequationes ita inter se cohaerent, ut altera ex altera per similem substitutionem producatur.

Corollarium 2.

1002. Si in aequatione transformata secundum substitutionem priorem ponatur $\frac{\partial z}{z} = Q\partial x + \frac{\partial v}{v}$, obtinebitur haec nova transformata

$$L\partial\partial v + \left(2LQ + \frac{\partial L}{\partial x} - M - \frac{L\partial N}{N\partial x}\right)\partial x\partial v$$
$$+ v\partial x\left(L\partial Q + LQQ\partial x + Q\partial L - MQ\partial x - \frac{LQ\partial N}{N} + N\partial x\right) = 0,$$

quae ergo ex ipsa proposita deducitur ponendo

$$\frac{\partial y}{y} = \frac{-Nv\partial x^2}{L(\partial v + Qv\partial x)}.$$

Scholion 1.

1003. Hinc combinando ambas substitutiones, quibus in binis praecedentibus problematibus sumus usi, substitutionem hujusmodi generalem adipiscimur

$$\frac{\partial y}{y} = \frac{P\partial z + Qz\partial x}{R\partial z + Sz\partial x}\cdot\partial x,$$

quae si in aequatione proposita

$$L\partial\partial y + M\partial x\partial y + Ny\partial x^2 = 0$$

substituatur, functiones P, Q, R, S ita definiri debent, ut in aequatione resultante variabilis z cum suis differentialibus nusquam plus

una dimensione teneat. Oriuntur autem termini quadrato ∂z^2 affecti, ad quos destruendos fieri oportet

$$L\partial x(PP+QR-PS)+L(R\partial P-P\partial R)+MPR\partial x+NRR\partial x=0,$$

seu $Q=\frac{PS}{R}-\frac{PP}{R}-\frac{\partial P}{\partial x}+\frac{P\partial R}{R\partial x}-\frac{MP}{L}-\frac{NR}{L}$;

tum vero pervenitur ad hanc aequationem

$$\left.\begin{array}{l} L\partial\partial z(PS-QR)+L\partial z(R\partial Q-Q\partial R+S\partial P-P\partial S) \\ \quad +\partial x\partial z[2LPQ+M(QR+PS)+2NRS] \\ \quad +Lz\partial x(S\partial Q-Q\partial S+QQ\partial x)+Sz\partial x^2(MQ+NS) \end{array}\right\}=0.$$

Verum facilius ad hanc aequationem generalem pervenitur, si ambae substitutiones alternatim in usum vocentur:

Scholion 2.

1004. Transformatio autem hic exposita eo magis est notatu digna, quod etiamsi aequatio transformata resolutionem admittat, inde tamen non nisi difficulter ipsa proposita resolvatur. Cum enim reperta fuerit functio ipsius x, quae loco z substituta aequationi transformatae satisfaciat, pro valore ipsius y inveniendo, insuper integrale hujus aequationis $\frac{\partial y}{y}=\frac{-Nz\partial x}{L\partial z}$ investigari oportet, ubi etsi variabiles x et y a se invicem sunt separatae, tamen difficultates insignes in ipsa integratione se exerere possunt. Fieri ergo poterit, ut ope hujus substitutionis, ejusmodi aequationum integralia exhiberi queant, quae directa via vix investigare liceat. Scilicet si eveniat, ut integrale aequationis transformatae vel ope methodi cujusdam supra expositae inveniri, vel per seriem abruptam exprimi possit, tum etiam ipsius aequationis propositae integrale habebitur. Etsi enim casu posteriori integrale tantum particulare innotescit, tamen ex eo semper in hoc aequationum genere integrale completum elici potest. Namque si aequationi

$$L\partial\partial y+M\partial x\partial y+Ny\partial x^2=0$$

particulariter satisfaciat valor $y = X$, ponatur $y = Xv$, fietque

$$\left.\begin{array}{r} LX\partial\partial v + 2L\partial X\partial v + Lv\partial\partial X \\ + MX\partial x\partial v + Mv\partial x\partial X \\ + NXv\partial x^2 \end{array}\right\} = 0.$$

At quia $X = y$ per hypothesin aequationi satisfacit, erit

$$L\partial\partial X + M\partial x\partial X + NX\partial x^2 = 0 \text{ et}$$

$$LX\partial\partial v + (2L\partial X + MX\partial x)\partial v = 0, \text{ seu}$$

$$\frac{\partial\partial v}{\partial v} + \frac{2\partial X}{X} + \frac{M\partial x}{L} = 0,$$

unde integrando oritur

$$XX\partial v = Ce^{-\int\frac{M\partial x}{L}}\partial x, \text{ porroque}$$

$$v = \int\frac{C\partial x}{XX}e^{-\int\frac{M\partial x}{L}};$$

ita ut integrale completum sit,

$$y = CX\int\frac{\partial x}{XX}e^{-\int\frac{M\partial x}{L}},$$

quod ergo ex quolibet integrali particulari $y = X$ elici potest.

Exemplum.

1005. *Aequationem differentio-differentialem*

$$xx(a + bx^n)\partial\partial y + x(c + ex^n)\partial x\partial y + fy\partial x^2 = 0$$

transformare ac per seriem integrare.

Cum hic sit $L = xx(a + bx^n)$, $M = x(c + ex^n)$ et $N = f$, utendum est hac substitutione

$$\frac{\partial y}{y} = \frac{-fz\partial x^2}{xx(a + bx^n)\partial z},$$

qua nostra aequatio reducitur ad hanc formam

$$xx(a+bx^n)\partial\partial z + x(2a - c + [(n+2)b - e]x^n)\partial x\partial z + fz\partial x^2 = 0,$$

pro cujus resolutione si ponatur

$$z = A x^{\lambda} + B x^{\lambda + n} + C x^{\lambda + 2n} + \text{etc.}$$

fieri debet

$$\lambda(\lambda - 1) a + \lambda(2a - c) + f = 0, \text{ seu}$$

$$\lambda\lambda a + \lambda(a - c) + f = 0, \text{ ergo } \lambda = \frac{-a + c \pm \sqrt{[(a-c)^2 - 4af]}}{2a}.$$

Series autem abrumpetur per 970, si haec expressio

$$-\frac{c}{2a} + \frac{e}{2b} - \frac{n}{2} \pm \left(\frac{e}{2b} - \frac{n}{2} - \frac{1}{2}\right) \pm \frac{\sqrt{[(a-c)^2 - 4af]}}{2a} = in,$$

denotante i numerum integrum positivum, hoc est

$$\text{vel } -\frac{c}{2a} + \frac{e}{b} - \frac{1}{2} - n \pm \frac{\sqrt{[(a-c)^2 - 4af]}}{2a} = in,$$

$$\text{vel } -\frac{c}{2a} + \frac{1}{2} \pm \frac{\sqrt{[(a-c)^2 - 4af]}}{2a} = in.$$

Sin autem ipsa aequatio proposita hoc modo in seriem resolvatur, haec abrumpetur, si fuerit

$$\frac{c}{2a} - \frac{e}{2b} \pm \frac{(b-e)}{2b} \pm \frac{\sqrt{[(a-c)^2 - 4af]}}{2a} = in,$$

hoc est

$$\text{vel } \frac{c}{2a} - \frac{e}{b} + \frac{1}{2} \pm \frac{\sqrt{[(a-c)^2 - 4af]}}{2a} = in$$

$$\text{vel } \frac{c}{2a} - \frac{1}{2} \pm \frac{\sqrt{[(a-c)^2 - 4af]}}{2a} = in.$$

Unde intelligitur, integrale finitum exhiberi posse, sive numerus integer i sit positivus sive negativus. Ad hanc vero duplicitatem jam prior substitutio perduxerat (998.), ita ut haec nova substitutio nullos novos casus integrabiles suppeditet.

Scholion 1.

1006. Ut tamen pateat, quomodo ex valore finito ipsius z valor finitus ipsius y elici queat, contemplemur casum

$$xx(a + bx^2)\partial\partial y + x(3a + ex^2)\partial x \partial y - 24ay\partial x^2 = 0,$$

ubi $n = 2$, $c = 3a$ et $f = -24a$, quae facta substitutione $\frac{\partial y}{y} = \frac{24az\partial x^2}{xx(a + bx^2)\partial z}$ abit in hanc

$$xx(a+bx^2)\partial\partial z + x[-a+(4b-e)xx]\partial x\partial z - 24 a z \partial x^2 = 0,$$

ubi pro serie ascendente fit

$$\lambda\lambda - 2\lambda - 24 = 0 \text{ vel } (\lambda - 6)(\lambda + 4) = 0.$$

Statuatur

$$z = A x^{-4} + B x^{-2} + C + D x^2 + \text{etc.}$$

erit

$$\left.\begin{array}{lllll} 20Aax^{-4} & +\ 6Bax^{-2} & * & +\ 2Dax^2 + \text{etc.} \\ & +20Ab & +6Bb & * \\ +\ 4Aa & +\ 2Ba & * & -\ 2Da \\ & -\ 4A(4b-e) & -\ 2B(4b-e) & * \\ -24Aa & -24Ba & -24Ca & -24Da \end{array}\right\} = 0.$$

Cum ergo sit $D = 0$, sequentes termini omnes tolluntur. Tum vero est

$$16Ba = 4A(b+e),\quad 24Ca = -2Bb + 2Be,$$

ergo

$$B = \frac{b+e}{4a},\quad C = \frac{e-b}{12a}B = \frac{ee-bb}{24aa}A,$$

hincque

$$z = A\left(\frac{1}{x^4} + \frac{b+e}{4axx} + \frac{ee-bb}{48aa}\right) = \frac{A[48aa + 12a(b+e)xx + (ee-bb)x^4]}{48aax^4},$$

unde sequitur

$$\partial z = A\partial x\left(\frac{-4}{x^5} - \frac{b-e}{2ax^3}\right) = \frac{-A\partial x}{2ax^5}[8a + (b+e)xx].$$

Ergo

$$\frac{\partial y}{y} = \frac{-[48aa + 12a(b+e)xx + (ee-bb)x^4]}{x(a+bx^2)[8a+(b+e)xx]}\partial x,$$

seu resolvendo

$$\frac{\partial y}{y} = \frac{-6\partial x}{x} + \frac{(5b-e)x\partial x}{a+bxx} + \frac{2(b+e)x\partial x}{8a+(b+e)xx},$$

hincque integrando

$$y = \frac{A}{x^6}(a+bxx)^{\frac{5b-e}{2b}}[8a+(b+e)xx].$$

Scholion 2.

1007. Quod hic casu fortuito evenisse videtur, ut ex valore ipsius z invento quantitas y commode definiri potuerit, idem perpetuo evenire oportere, sequenti modo in genere ostendi potest. Cum enim aequatio proposita

$$L\partial\partial y + M\partial x\partial y + Ny\partial x^2 = 0$$

ope substitutionis $\frac{\partial y}{y} = -\frac{Nz\partial x^2}{L\partial z}$ in hanc sit transformata

$$L\partial\partial z - M\partial x\partial z - \frac{L\partial N\partial z}{N} + \partial L\partial z + Nz\partial x^2 = 0,$$

si haec per $L\partial z$ dividatur, prodit

$$\frac{\partial\partial z}{\partial z} - \frac{M\partial x}{L} - \frac{\partial N}{N} + \frac{\partial L}{L} = -\frac{Nz\partial x^2}{L\partial z} = \frac{\partial y}{y},$$

ex qua integrando elicitur

$$y = \frac{aL\partial z}{N\partial x} e^{-\int\frac{M\partial x}{L}},$$

quae invento valore ipsius z, statim sine ulteriori integratione praebet valorem ipsius y.

Cum porro sit

$$\partial y = \frac{-Nyz\partial x^2}{L\partial z}, \text{ erit } \partial y = -az\partial x \,.\, e^{-\int\frac{M\partial x}{L}},$$

hincque

$$y\partial y = \frac{-aaLz\partial z}{N} e^{-2\int\frac{M\partial x}{L}},$$

atque hae relationes eo magis sunt notatu dignae, quod ex iis aequatio proposita nonnisi per plures ambages ad transformatam reduci possit. Ipsa enim formula pro y substituta perducit ad aequationem differentialem tertii gradus, quae autem manifesto integrationem admittit, ipsamque aequationem hic inventam suppeditat. Hinc igitur occasionem adipiscimur ejusmodi substitutiones investi-

gandi, quae quidem ad differentialia tertii gradus ascendant, verum tamen per integrationem ad differentialia secunda redigi se patiantur.

Problema 127.

1008. Aequationem differentio-differentialem

$$L\partial\partial y + M\partial x\partial y + Ny\partial x^2 = 0$$

ope hujusmodi substitutionis $y = \frac{P\partial z}{\partial x}$ in aliam aequationem pariter differentio-differentialem transformare.

Solutio.

Ob $y = \frac{P\partial z}{\partial x}$, fit

$$\partial y = \frac{P\partial\partial z + \partial P\partial z}{\partial x} \text{ et } \partial\partial y = \frac{P\partial^3 z + 2\partial P\partial\partial z + \partial z\partial\partial P}{\partial x},$$

quibus formulis substitutis oritur haec aequatio differentialis tertii gradus

$$LP\partial^3 z + 2L\partial P\partial\partial z + L\partial z\partial\partial P + MP\partial x\partial\partial z$$
$$+ M\partial x\partial P\partial z + NP\partial x^2\partial z = 0,$$

quam ita comparatam assumamus, ut per functionem ipsius x, quae sit Q, multiplicata integrabilis evadat. Integrabilis ergo sit haec forma

$$LPQ\partial^3 z + 2LQ\partial P\partial\partial z + MPQ\partial x\partial\partial z + LQ\partial z\partial\partial P$$
$$+ MQ\partial x\partial P\partial z + NPQ\partial x^2\partial z = 0,$$

cujus integrale sit

$$LPQ\partial\partial z + S\partial x\partial z + Tz\partial x^2 = C\partial x^2,$$

unde colligitur

$$\partial\partial z(2LQ\partial P + MPQ\partial x) = \partial\partial z(\partial.LPQ + S\partial x),$$
$$\partial z(LQ\partial\partial P + MQ\partial x\partial P + NPQ\partial x^2) = \partial z(\partial x\partial S + T\partial x^2),$$

et $z\partial x^2\partial T = 0$, ideoque T quantitas constans.

Inde autem fit

$$S\partial x = LQ\partial P - LP\partial Q - PQ\partial L + MPQ\partial x,$$

ex quo per alteram conditionem elicitur

$$\begin{aligned}T\partial x^2 = {} & LQ\partial\partial P + MQ\partial x\partial P + NPQ\partial x^2 - LQ\partial\partial P - L\partial P\partial Q - Q\partial P\partial L \\ & + LP\partial\partial Q + L\partial P\partial Q + P\partial Q\partial L + PQ\partial\partial L + P\partial Q\partial L + Q\partial P\partial L \\ & - MP\partial x\partial Q - MQ\partial x\partial P - PQ\partial x\partial M,\ \text{sive}\end{aligned}$$

$$T\partial x^2 = P\partial\partial.LQ - P\partial x\partial.MQ + PNQ\partial x^2.$$

Quare cum T sit quantitas constans, ponatur $T = a$, atque hinc commode functio P definitur, scilicet

$$P = \frac{a\partial x^2}{\partial\partial.LQ - \partial x\partial.MQ + NQ\partial x^2},$$

hocque valore pro P assumto, aequatio proposita ope substitutionis $y = \frac{P\partial z}{\partial x}$ transformatur in hanc

$$\begin{aligned}LPQ\partial\partial z + \partial z(LQ\partial P - LP\partial Q - PQ\partial L + MPQ\partial x) \\ + az\partial x^2 = C\partial x^2,\end{aligned}$$

ubi cum z constante quantitate augere liceat, constans C omitti potest. Dividatur ergo haec aequatio per PQ et prodibit

$$L\partial\partial z + \partial z\left(\frac{L\partial P}{P} - \frac{L\partial Q}{Q} - \partial L + M\partial x\right) + \frac{az\partial x^2}{PQ} = 0,$$

seu in postremo termino valorem ipsius P substituendo

$$\begin{aligned}L\partial\partial z + \partial z\left(\frac{L\partial P}{P} - \frac{\partial.LQ}{Q} + M\partial x\right) \\ + \frac{z}{Q}(\partial\partial.LQ - \partial x\partial.MQ + NQ\partial x^2) = 0,\end{aligned}$$

atque hic pro Q functionem quamcunque ipsius x accipere licet.

Corollarium 1.

1009. Hinc praecedens substitutio derivatur ponendo

$$\partial\partial.LQ - \partial x\partial.MQ = 0,\ \text{ideoque}$$
$$\partial.LQ - MQ\partial x = C\partial x,\ \text{seu}$$
$$e^{-\int\frac{M\partial x}{L}}LQ = C\int e^{-\int\frac{M\partial x}{L}}\partial x + D.$$

Namque si hic capiatur $C = 0$, erit

$$Q = \frac{D}{L} e^{+\int \frac{M \partial x}{L}}, \text{ et } P = \frac{\alpha \partial x^2}{NQ \partial x^2} = \frac{\alpha}{NQ}, \text{ seu}$$

$$P = \frac{\alpha L}{N} e^{-\int \frac{M \partial x}{L}}, \text{ ut ante.}$$

Corollarium 2.

1010. Sin autem ponamus

$\partial \partial . LQ - \partial x \partial . MQ = \partial X \partial x$, ut sit

$P = \frac{\alpha \partial x}{\partial X + NQ \partial x}$, erit

$\partial . LQ - MQ \partial x = X \partial x + A \partial x$,

porroque integrando

$$e^{-\int \frac{M \partial x}{L}} LQ = \int e^{-\int \frac{M \partial x}{L}} \partial x (X + A) + B \text{ et}$$

$$Q = \frac{1}{L} e^{\int \frac{M \partial x}{L}} \int e^{-\int \frac{M \partial x}{L}} \partial x (X + A) + \frac{B}{L} e^{\int \frac{M \partial x}{L}}.$$

Corollarium 3.

1011. Ponatur $\int e^{-\int \frac{M \partial x}{L}} X \partial x = e^{-\int \frac{M \partial x}{L}} V$, et $A = 0$, $B = 0$, erit $X = \frac{\partial V}{\partial x} - \frac{MV}{L}$ et $Q = \frac{V}{L}$, ideoque

$$P = \frac{\alpha \partial x}{\frac{\partial \partial V}{\partial x} - \frac{M}{L} \partial V - V \partial . \frac{M}{L} + \frac{N}{L} V \partial x}.$$

Si igitur sit $V = \alpha$, erit $Q = \frac{\alpha}{L}$,

$P = \frac{LL \partial x}{LN \partial x - L \partial M + M \partial L}$,

et aequatio resultans

$$L \partial \partial z + \partial z \left(\frac{L \partial P}{P} + M \partial x\right) + \frac{z \partial x (LN \partial x - L \partial M + M \partial L)}{L} = 0.$$

Scholion.

1012. Haec autem nimis sunt generalia, quam ut inde quicquam ad usum communem concludi possit. Utcunque autem trans-

formatio instituatur, et aequatio transformata in seriem resolvatur, haec nullis aliis casibus abrumpi videtur, nisi iis quibus ipsa aequatio proposita, et inde per primam substitutionem transformata, ad seriem alicubi terminatam reducitur. Ex quo perspicuum est ope hujusmodi transformationum vix unquam novos casus integrabiles erui posse. Verum dum hactenus loco variabilis y aliam z per substitutionem introduximus, altera x, ex cujus potestatibus series formabantur, retenta, nunc etiam paucis exploremus, quomodo loco ipsius x aliam variabilem t introducendo, transformationem perfici oporteat; ubi imprimis notetur necesse est, cum ante elementum ∂x assumtum fuerit constans, jam in transformata elementum ∂t constans accipi debere. Hic igitur t scribetur loco certae cujuspiam functionis ipsius x, quam autem ita comparatam esse decet, ut aequatio resultans ne nimis fiat complicata.

Problema 128.

1013. Proposita aequatione differentio-differentiali

$$L\partial\partial y + M\partial x\partial y + Ny\partial x^2 = 0,$$

loco quantitatis x aliam t introducere, quae functioni cuipiam ipsius x aequetur.

Solutio.

Divisa aequatione per ∂x, reqraesentetur aequatio ita

$$L\partial.\frac{\partial y}{\partial x} + M\partial y + Ny\partial x = 0,$$

ut jam consideratio elementi ∂x, quod constans erat assumtum, sit exclusa. Cum t aequetur functioni cuipiam ipsius x, fiat inde $\partial t = P\partial x$, seu $\partial x = \frac{\partial t}{P}$, unde nanciscimur

$$L\partial.\frac{P\partial y}{\partial t} + M\partial y + \frac{Ny\partial t}{P} = 0,$$

ac sumto elemento ∂t constante

$$LP\partial\partial y + L\partial P\partial y + M\partial t\partial y + \frac{Ny\partial t^2}{P} = 0,$$

ubi tantum superest, ut in quantitatibus finitis, quae adhuc variabilem x complectuntur, ejus loco altera t introducatur.

Exemplum.

1014. *Proposita sit haec aequatio*

$$xx(a+bx^n)\partial\partial y+x(c+ex^n)\partial x\partial y+(f+gx^n)y\partial x^2=0,$$

in quam loco formulae $h+kx^n$ *introducatur* t.

Cum ergo sit

$$t=h+kx^n, \text{ erit } \partial t=nkx^{n-1}\partial x,$$

ideoque

$$P=nkx^{n-1} \text{ et } \partial P=n(n-1)kx^{n-2}\partial x=\frac{(n-1)\partial t}{x}.$$

Quare habebimus

$$nkx^{n+1}(a+bx^n)\partial\partial y+(n-1)x\partial t\partial y(a+bx^n)+x(c+ex^n)\partial t\partial y$$
$$+\frac{(f+gx^n)y\partial t^2}{nkx^{n-1}}=0$$

sive

$$nk(a+bx^n)\partial\partial y+\frac{(n-1)\partial t\partial y(a+bx^n)+\partial t\partial y(c+ex^n)}{x^n}$$
$$+\frac{(f+gx^n)y\partial t^2}{nkx^{2n}}=0.$$

Nunc vero est $x^n=\frac{t-h}{k}$, qui valor substitutus praebet

$$n(ak-bh+bt)\partial\partial y+\frac{(n-1)\partial t\partial y(ak-bh+bt)+\partial t\partial y(ck-eh+et)}{t-h}$$
$$+\frac{(fk-gh+gt)y\partial t^2}{n(t-h)^2}=0.$$

Verum hic ita ubique $t-h$ occurrit, ut aequatio simplicior evadat loco $t-h$ scribendo u, tum autem perinde est, ac si loco

potestatis x^n scripsissemus quantitatem u: neque ergo hinc quicquam lucri pro novis seriebus eruendis redundat.

Corollarium.

1015. Si in aequatione generali loco x^m scribere velimus t, erit

$$\partial t = m x^{m-1} \partial x \text{ et } P = m x^{m-1},$$

et aequatio resultabit, ob

$$\partial P = m(m-1) x^{m-2} \partial x = \frac{(m-1)\partial t}{x},$$

ista

$$m L x^{m-1} \partial\partial y + \frac{(m-1) L \partial t \partial y}{x} + M \partial t \partial y + \frac{N y \partial t^2}{m x^{m-1}} = 0$$

seu

$$m L \partial\partial y + \frac{(m-1) L \partial t \partial y}{t} + \frac{M x \partial t \partial y}{t} + \frac{N x x y \partial t^2}{m t t} = 0.$$

Scholion.

1016. Plura de hujusmodi aequationum transformationibus tradere haud necesse videtur, cum ex his fontibus haud difficulter omnes transformationes ad usum idoneae derivari queant. Datur autem alia methodus prorsus singularis hujusmodi aequationum differentio-differentialium integralia exprimendi, quae per formulas integrales binas variabiles involventes expeditur, dum altera in integratione ut constans tractatur. Ita si P fuerit functio quaecunque binarum variabilium x et u, ac ponatur $y = \int P \partial x$, considerando u in integratione ut constantem, integrale hoc $\int P \partial x$ erit functio ipsarum x et u, quod ita determinatum, ut evanescat posito $x = 0$, si deinceps statuatur $x = a$, obtinebitur functio ipsius u ipsi y aequalis, quae si satisfaciat aequationi cuipiam differentiali inter u et y propositae, haec aequatio resolvetur formula $y = \int P \partial x$, quae ut ejus integrale spectari poterit. Atque hoc modo innume-

rabilium aequationum differentio-differentialium integralia exhiberi possunt, quae aliis methodis prorsus intractabiles videntur. Quamquam autem formula $\int P \partial x$, spectata quantitate u ut constante, actu integrari nequit, tamen ejus integrale in hoc negotio pro cognito accipi potest, quia ejus valor saltem per approximationes assignari potest. Scilicet dum sumta x pro abscissa, si P denotet applicatam orthogonalem ei convenientem, formula $\int P \partial x$ exprimet aream ejusdem curvae abcissae x insistentem, ac posito $x = a$, area habetur determinata valori $y = \int P \partial x$, prouti cum modo definivimus, aequalis, quae ergo, uti loqui solent, per quadraturas curuarum assignari potest, ex quo haec integrandi ratio commode appellatur constructio per quadraturas. Hic autem imprimis ad eam rationem, qua integralia in particularia et completa distinximus, attendi conveniet; unde sollicite est cavendum, ne integralia hoc modo inventa pro completis habeantur, nisi quatenus binas constantes arbitrarias involvant. Cum igitur eidem aequationi differentiali infinita integralia particularia conveniant, mirandum non est, si hoc modo pro eadem aequatione proposita plura integralia diversa inveniamus. Hoc autem argumentum fere prorsus est novum, neque a quoquam adhuc pertractatum, si quidem nonnulla specimina, quae equidem jam dudum dedi, excipiantur; ex quo dubitare non licet, quin ista methodus, si diligentius excolatur, aliquando forte praeclara incrementa in Analysin sit allatura.

CAPUT X.

DE CONSTRUCTIONE AEQUATIONUM DIFFERENTIO-DIFFERENTIALIUM PER QUADRATURAS CURVARUM.

Problema 125.

1017.

Si fuerit $y = \int V \partial x$, denotante V functionem quamcumque binarum quantitatum x et u, quarum autem haec u in integratione ut constans spectatur, post integrationem vero statuatur $x = a$, ut y aequetur functioni cuidam ipsius u; quodsi jam u variabilis sumatur, investigare valorem ipsius $\frac{\partial y}{\partial u}$.

Solutio.

Cum $\int V \partial x$ exhibeat functionem quandam binarum quantitatum x et u, cujus differentiale, sumta u constante, est $= V \partial x$, si tam u quam x ut variabiles tractentur, differentiale aequationis $y = \int V \partial x$ talem habebit formam, $\partial y = V \partial x + U \partial u$, quae, quia est differentiale verum, necesse est sit $\left(\frac{\partial V}{\partial u}\right) = \left(\frac{\partial U}{\partial x}\right)$. At cum V sit functio data ipsarum x et u, ponatur $\partial V = P \partial x + Q \partial u$, eritque $\left(\frac{\partial V}{\partial u}\right) = Q$, ideoque $\left(\frac{\partial U}{\partial x}\right) = Q$. Hinc considerata iterum u ut constante, erit $\partial U = Q \partial x$, et $U = \int Q \partial x$, in qua integratione sola x pro variabili habetur. Quocirca si hunc valorem $\int Q \partial x$ ut cognitum spectemus, quippe quem per quadraturas assignare licet,

erit $\partial y = V \partial x + \partial u \int Q \partial x$. Quaerimus autem id ipsius y differentiale, quod ex variabilitate ipsius u tantum nascitur; quod cum sit $\partial y = \partial u \int Q \partial x$ erit valor quaesitus $\frac{\partial y}{\partial u} = \int Q \partial x$, si nempe post integrationem itidem ponatur $x = a$.

Corollarium 1.

1018. Cum sit $y = \int V \partial x$ functio ipsarum x et u, per integrationem autem formulae $V \partial x$, in qua u constans spectatur, functio quaecunque ipsius u loco constantis accedere possit, functio y per se erit indeterminata, determinabitur autem statim, atque integrale $\int V \partial x$ ita accipiatur, ut evanescat posito $x = 0$.

Corollarium 2.

1019. Hac conditione observata evanescet y posito $x = 0$, quicunque valor alteri quantitati u tribuatur, erit ergo etiam $y + \partial u \left(\frac{\partial y}{\partial u}\right) = 0$ facto $x = 0$, ergo etiam $\left(\frac{\partial y}{\partial u}\right) = 0$. Unde patet $\int Q \partial x = \frac{\partial y}{\partial u}$ ita quoque accipi debere, ut posito $x = 0$ evanescat.

Corollarium 3.

1020. Cum $y = \int V \partial x$ erit $\left(\frac{\partial y}{\partial x}\right) = V$, hinc

$$\left(\frac{\partial \partial y}{\partial u \partial x}\right) = \left(\frac{\partial V}{\partial u}\right).$$

At si ponatur $\left(\frac{\partial y}{\partial u}\right) = Z$, erit quoque

$$\left(\frac{\partial \partial y}{\partial u \partial x}\right) = \left(\frac{\partial Z}{\partial x}\right), \text{ ergo } \left(\frac{\partial Z}{\partial x}\right) = \left(\frac{\partial V}{\partial u}\right).$$

Quare spectata u ut constante, erit

$$\partial Z = \partial x \left(\frac{\partial V}{\partial u}\right), \text{ et } Z = \int \partial x \left(\frac{\partial V}{\partial u}\right)$$

ideoque

$$\left(\frac{\partial y}{\partial u}\right) = \int \partial x \left(\frac{\partial V}{\partial u}\right).$$

Corollarium 4.

1021. Quodsi ergo post integrationes ita absolutas, ut integralia evanescant posito $x = 0$, ponatur $x = a$, tam valor $y = \int V \partial x$ quam $\frac{\partial y}{\partial u} - \int \partial x \left(\frac{\partial V}{\partial u}\right)$ erit functio determinata ipsius u.

Corollarium 5.

1022. Simili modo ulterius progrediendo erit

$$\frac{\partial \partial y}{\partial u^2} = \int \partial x \left(\frac{\partial \partial V}{\partial u^2}\right).$$

Quare si L, M et N denotent functiones quascunque ipsius u, erit

$$\frac{L \partial \partial y}{\partial u^2} + \frac{M \partial y}{\partial u} + N y = \int \partial x \left[L \left(\frac{\partial \partial V}{\partial u^2}\right) + M \left(\frac{\partial V}{\partial u}\right) + N V\right]$$

totumque negotium huc redit, ut ista formula integrationem admittat.

Scholion.

1023. Datis scilicet ipsius u functionibus L, M, N, quaeri debet functio V binarum variabilium x et u, ita ut spectata u constante formula

$$\left[L \left(\frac{\partial \partial V}{\partial u^2}\right) + M \left(\frac{\partial V}{\partial u}\right) + N V\right] \partial x$$

absolute fiat integrabilis, cujus integrale, ut sit determinatum ita capiatur, ut posito $x = 0$, evanescat. Tum vero statuatur $x = a$, ac si illud integrale etiam hoc casu evanescat, erit

$$\frac{L \partial \partial y}{\partial u^2} + \frac{M \partial y}{\partial u} + N y = 0,$$

hineque aequationi satisfacit valor $y = \int V \partial x$, lege indicata sumtus. Problema autem, datis functionibus L, M et N, investigandi functionem V maxime est indeterminatum, neque methodis adhuc cognitis in genere resolvi potest; ex quo conveniet id inverso modo tractari, ut sumta functione V, alterae L, M et N indagentur. Hinc aequationes differentio-differentiales consequemur, quarum inte-

gralia hoc modo assignare valebimus, quae si aliis methodis tractari nequeant, insigne lucrum suppeditant. Quodsi integrale illud

$$\int [L (\frac{\partial \partial V}{\partial u^2}) + M (\frac{\partial V}{\partial u}) + N V] \partial x$$

posito $x = a$ non evanescat, sed datam ipsius u functionem U exhibeat, valor $y = \int V \partial x$ conveniet huic aequationi

$$\frac{L \partial \partial y}{\partial u^2} + \frac{M \partial y}{\partial u} + N y = U,$$

quae cum infinitis modis in alias formas transmutari possit, etiam harum integralia innotescent, ubi simul hoc commode evenit, ut etiamsi integrale tantum particulare obtineatur, inde tamen plerumque integrale completum haud difficulter colligi queat.

Problema 130.

1024. Invenire aequationes differentio-differentiales formae $\frac{L \partial \partial y}{\partial u^2} + \frac{M \partial y}{\partial u} + N y = U$, ut L, M, N et U sint functiones ipsius u, cujus elementum ∂u hic pro constante accipitur, quarum integrale ope constructionis per quadraturas exhiberi possit.

Solutio.

Sumatur functio quaecunque binarum variabilium u et x, quae sit V, capiaturque integrale $\int V \partial x$ spectata quantitate u ut constante, ita ut posito $x = 0$, evanescat, tum vero fiat $x = a$, denotante a quantitatem quamcunque constantem, ut jam $\int V \partial x$ exprimat functionem quandam ipsius u tantum, cui quantitas y aequetur, ut sit $y = \int V \partial x$. Cum jam sit

$$\frac{\partial y}{\partial u} = \int \partial x (\frac{\partial V}{\partial u}), \text{ et } \frac{\partial \partial y}{\partial u^2} = \int \partial x (\frac{\partial \partial V}{\partial u^2}),$$

his integralibus pariter ita sumtis, ut posito $x = 0$ evanescant, tum vero statuatur $x = a$, quaerantur functiones L, M, N ipsius u, ut haec formula

$$\int \partial x [L (\frac{\partial \partial V}{\partial u^2}) + M (\frac{\partial V}{\partial u}) + N V]$$

fiat absolute integrabilis, ejusque integrale ita determinetur, ut posito $x = a$, fiat id $= U$. Quod si fuerit praestitum, evidens est, aequationi differentio - differentiali

$$\frac{L \partial \partial y}{\partial u^2} + \frac{M \partial y}{\partial u} + N y = U$$

satisfacere formulam assumtam $y = \int V \partial x$.

Corollarium 1.

1025. Assumtio ergo functionis V non penitus arbitrio nostro permittitur, sed ad hoc potissimum est spectandum, ut talis forma

$$\int \partial x \left[L \left(\frac{\partial \partial V}{\partial u^2}\right) + M \left(\frac{\partial V}{\partial u}\right) + N V\right],$$

per se fiat integrabilis.

Corollarium 2.

1026. Infinitae ergo hinc statim excluduntur formae ad hunc scopum ineptae, cujusmodi sunt $V = UP$, existente U functione ipsius u et P ipsius x tantum; quia tum foret

$$y = U \int P \partial y, \quad \frac{\partial y}{\partial u} = \frac{\partial U}{\partial u} \int P \partial x \quad \text{et} \quad \frac{\partial \partial y}{\partial u^2} = \frac{\partial \partial U}{\partial u^2} \int P \partial x,$$

quippe quae idem integrale complectuntur, ita ut ex earum conjunctione formula absolute integrabilis confici nequeat.

Exemplum 1.

1027. *Sit* $V = x^n \sqrt{\frac{uu + xx}{cc - xx}}$ *et* $y = \int x^n \partial x \sqrt{\frac{uu + xx}{cc - xx}}$ *integrali evanescente posito* $x = 0$, *tum vero facto* $x = a$.

Erit ergo

$$\left(\frac{\partial V}{\partial u}\right) = x^n \frac{u}{\sqrt{(uu + xx)(cc - xx)}} \text{ et } \left(\frac{\partial \partial V}{\partial u^2}\right) = x^n \frac{xx}{(uu + xx)^{\frac{3}{2}} \sqrt{(cc - xx)}},$$

et integrabilis reddi debet haec formula

$$x^n \partial x \left(\frac{Lxx}{(uu+xx)^{\frac{3}{2}} \sqrt{(cc-xx)}} + \frac{Mu}{\sqrt{(uu+xx)(cc-xx)}} + N \sqrt{\frac{uu+xx}{cc-xx}} \right),$$

seu

$$\frac{x^n \partial x}{(uu+xx)^{\frac{3}{2}} \sqrt{(cc-xx)}} [Lxx + Mu(uu+xx) + N(uu+xx)^2].$$

Statuatur integrale $= \frac{x^{n+1}\sqrt{(cc-xx)}}{\sqrt{(uu+xx)}}$, cujus differentiale cum sit

$$\frac{(n+1)x^n(cc-xx)(uu+xx) - x^{n+2}(uu+xx) - x^{n+2}(cc-xx)}{(uu+xx)^{\frac{3}{2}} \sqrt{(cc-xx)}} \cdot \partial x,$$

seu

$$\frac{x^n \partial x}{(uu+xx)^{\frac{3}{2}} \sqrt{(cc-xx)}} \left\{ \begin{matrix} (n+1)ccuu + (n+1)ccxx - (n+1)uuxx - (n+1)x^4 \\ -ccxx \qquad -uuxx \end{matrix} \right\},$$

cum qua si proposita comparetur, fiet

$$Mu^3 + Nu^4 = (n+1)ccuu,$$

$$L + Mu + 2Nuu = ncc - (n+2)uu \text{ et}$$

$$N = -(n+1).$$

Hinc elicitur

$$Mu = (n+1)(cc+uu), \quad \text{seu } M = \frac{(n+1)(cc+uu)}{u}, \text{ et}$$

$$L = -(n+1)(cc+uu) + 2(n+1)uu + ncc - (n+2)uu,$$

seu $L = -cc - uu$.

Quamobrem habebimus

$$\frac{(cc+uu)\partial\partial y}{\partial u^2} + \frac{(n+1)(cc+uu)\partial y}{u\partial u} - (n+1)y = \frac{a^{n+1}\sqrt{(cc-aa)}}{\sqrt{(aa+uu)}},$$

**

cui aequationi satisfacit $y = \int x^n \partial x \sqrt{\frac{uu+xx}{cc-xx}}$, integratione absoluta ut est indicatum.

Corollarium 1.

1028. Sumtò ergo $a = c$, formula integralis

$$y = \int x^n \partial x \sqrt{\frac{uu+xx}{cc-xx}}$$

posito post integrationem $x = c$, exhibebit integrale hujus aequationis

$$u(cc+uu)\partial\partial y - (n+1)(cc+uu)\partial u \partial y + (n+1) u y \partial u^2 = 0,$$

seu

$$\partial\partial y - \frac{(n+1)\partial u \partial y}{u} + \frac{(n+1) y \partial u^2}{cc+uu} = 0.$$

Corollarium 2.

1029. Si sit $n = 1$, per integrationem invenitur

$$\int x \partial x \sqrt{\frac{uu+xx}{cc-xx}} = \tfrac{1}{4}(cc+uu) \text{ Ang. sin. } \frac{2xx-cc+uu}{cc+uu}$$
$$- \tfrac{1}{2}\sqrt{(ccuu + ccxx - uuxx - x^4)},$$
$$- \tfrac{1}{4}(cc+uu) \text{ Ang. sin. } \frac{-cc+uu}{cc+uu} + \tfrac{1}{2} cu,$$

et posito $x = c$ fit

$$y = \tfrac{1}{4}(cc+uu) \text{ Ang. cos. } \frac{uu-cc}{cc+uu} + \tfrac{1}{2} cu, \text{ hincque}$$
$$\frac{\partial y}{\partial u} = \tfrac{1}{2} u \text{ Ang. cos. } \frac{uu-cc}{cc+uu} \text{ et}$$
$$\frac{\partial\partial y}{\partial u^2} = \tfrac{1}{2} \text{ Ang. cos. } \frac{uu-cc}{cc+uu} - \frac{cu}{cc+uu},$$

quae formulae evidenter satisfaciunt aequationi

$$\partial\partial y - \frac{2\partial u \partial y}{u} + \frac{2 y \partial u^2}{cc+uu} = 0.$$

Corollarium 3.

1030. Hoc casu integrale etiam hoc modo exprimi potest

$$y = \tfrac{1}{4}(cc+uu) \text{ Ang. sin. } \frac{2cu}{cc+uu} + \tfrac{1}{2} cu,$$

seu cum ejus multiplum quodvis aeque satisfaciat

$$y = (cc + uu)\ \text{Ang. sin.}\ \frac{2cu}{cc+uu} + 2cu,$$

satisfacit vero etiam $y = cc + uu$, unde integrale completum est

$$y = \alpha (cc + uu)\ \text{Ang. sin.}\ \frac{2cu}{cc+uu} + 2\alpha cu + \beta (cc + uu).$$

Scholion.

1031. Quod valor $y = cc + uu$ satisfaciat, ex integrali invento concludere licet, quia enim Ang. sin. $\frac{2cu}{cc+uu}$ est functio multiplex et termino 2π augeri potest, integrale ipsium augeri potest termino $2\pi(cc + uu)$. At in genere differentia binorum integralium quoque satisfacit, ergo etiam satisfacere debet $y = 2\pi(cc + uu)$ et generatim $y = \beta(cc + uu)$. Ex hoc casu facilius perspicitur, quomodo valor assumtus aequationi generali satisfaciat, etiamsi is per integrationem evolvi nequeat. Patet autem $n + 1$ esse debere numerum positivum, quia alioquin conditio integralis, ut posito $x = 0$ evanescat, impleri nequit.

Exemplum 2.

1032. *Sumatur*

$$V = x^{n-1} (uu + xx)^{\mu} (cc - xx)^{\nu}, \text{ erit}$$

$$\left(\frac{\partial V}{\partial u}\right) = 2\mu u x^{n-1} (uu + xx)^{\mu-1} (cc - xx)^{\nu} \text{ et}$$

$$\left(\frac{\partial\partial V}{\partial u^2}\right) = 2\mu x^{n-1} (cc-xx)^{\nu} [(uu+xx)^{\mu-1} + 2(\mu-1)uu(uu+xx)^{\mu-2}]$$

$$\text{seu} = 2\mu x^{n-1} (cc - xx)^{\nu} (uu+xx)^{\mu-2} [(2\mu - 1)uu + xx].$$

Integrabilis igitur reddi debet absolute haec formula

$$\int x^{n-1} \partial x (cc - xx)^{\nu} (uu + xx)^{\mu-2} \times$$

$$(2\mu[(2\mu - 1)uu + xx]L + 2\mu u(uu + xx)M + (uu + xx)^2 N),$$

seu

$$\int x^{n-1}\partial x(cc-xx)^{\nu}(uu+xx)^{\mu-2}\left\{\begin{array}{lll}2\mu(2\mu-1)Luu & +2\mu Lxx & +Nx^4\\ +2\mu Mu^3 & +2\mu Muxx & \\ +Nu^4 & +2Nuuxx & \end{array}\right\}.$$

Statuatur integrale. $x^n(uu+xx)^{\mu-1}(cc-xx)^{\nu+1}$, cujus differentiale cum sit

$$x^{n-1}\partial x(uu+xx)^{\mu-2}(cc-xx)^{\nu}\times$$
$$[n(uu+xx)(cc-xx)+2(\mu-1)xx(cc-xx)-2(\nu+1)xx(uu+xx)],$$

erit

$$2\mu(2\mu-1)Luu+2\mu Mu^3+Nu^4=nccuu,$$
$$2\mu L+2\mu Mu+2Nuu=ncc-nuu+2(\mu-1)cc-2(\nu+1)uu,$$
$$N=-n-2(\mu-1)-2(\nu+1)=-n-2\mu-2\nu$$

At prima

$$2\mu(2\mu-1)L+2\mu Mu+Nuu=ncc$$

demta secunda dat

$$4\mu(\mu-1)L-Nuu=(n+2\nu+2)uu-2(\mu-1)cc, \text{ seu}$$
$$4\mu(\mu-1)L=-2(\mu-1)(uu+cc), \text{ hinc } L=\frac{-cc-uu}{2\mu},$$

qui valor in prima sustitutus dat

$$-(2\mu-1)(cc+uu)+2\mu Mu-(n+2\mu+2\nu)uu=ncc,$$

seu

$$2\mu Mu=(n+2\mu-1)cc+(n+4\mu+2\nu-1)uu.$$

Ergo

$$M=\frac{(n+2\mu-1)(cc+uu)}{2\mu u}+\frac{(\mu+\nu)}{\mu}u.$$

Si $n>0$, superius integrale evanescit posito $x=0$, quare si ponamus $x=a$, orietur haec aequatio

$$-\frac{(cc+uu)\partial\partial y}{2\mu\partial u^2}+\frac{(n+2\mu-1)(cc+uu)\partial y}{2\mu u\partial u}+\frac{(\mu+\nu)u\partial y}{\mu\partial u}$$
$$-(n+2\mu+2\nu)y=a^n(aa+uu)^{\mu-1}(cc-aa)^{\nu+1},$$

cujus integrale est

$$y=\int x^{n-1}\partial x(uu+xx)^{\mu}(cc-xx)^{\nu},$$

integrali hoc ita sumto ut evanescat posito $x=0$; tum vero posito $x=a$.

Corollarium 1.

1033. Si capiatur $a = c$, ut postrema pars fiat $= 0$, siquidem exponens $\nu + 1$ sit nihilo major, formula

$$y = \int x^{n-1}\partial x (uu + xx)^{\mu} (cc - xx)^{\nu}$$

posito $x = c$, post integrationem ita peractam ut casu $x = 0$ fiat $y = 0$, erit integrale hujus aequationis

$$u(cc+uu)\partial\partial y-(n+2\mu-1)(cc+uu)\partial u\partial y-2(\mu+\nu)uu\partial u\partial y$$
$$+2\mu(n+2\mu+2\nu)uy\partial u^2 = 0.$$

Corollarium 2.

1034. Sit $n+2\mu-1 = \alpha$, et $n+4\mu+2\nu-1 = \beta$, fiet $2\mu = \alpha+1-n$ et $2\nu = \beta+1-n-2\alpha-2+2n = \beta-1+n-2\alpha$, et aequationis

$$u(cc+uu)\partial\partial y-(\alpha cc+\beta uu)\partial u\partial y+(\alpha+1-n)(\beta-\alpha+n)uy\partial u^2 = 0$$

integrale erit

$$y = \int x^{n-1}\partial x (uu+xx)^{\frac{\alpha+1-n}{2}}(cc-xx)^{\frac{\beta-1+n-2\alpha}{2}},$$

posito $x = c$, si modo sit $n > 0$ et $\beta - 1 + n > 2\alpha$.

Scholion.

1035. Haec constructio latissime ad hanc aequationem patet

$$xx(a+bx^n)\partial\partial z+x(c+ex^n)\partial x\partial z+(f+gx^n)z\partial x^2 = 0,$$

primo enim hic sine detrimento amplitudinis sumi potest $n = 2$, ponendo $x^n = uu$. Tum vero uti supra §. 997. vidimus, ponendo

$$z = x^{\frac{a-c}{a}+h}(a+bx)^{\frac{bc-ae}{nab}+1}y,$$

aequatio abit in hanc

$$xx(a+bx^n)\partial\partial y+x[2a-c+2ah+(2b-e+2nb+2bh)x^n]\partial x\partial y$$
$$+[f+ah-ch+ahh+(g+(b-e+nb+bh)(n+h))x^n]y\partial x^2 = 0,$$

ubi si h ita accipiatur, ut sit $ahh + (a - c)h + f = 0$, prodit aequatio formae, cujus constructionem dedimus. In casibus autem specialibus difficultates occurrere possunt, quibus superandis sequentia exempla inserviunt.

Exemplum 3.

1036. *Sit* $V = e^{mux} x^n (c - x)^\nu$, *erit*

$$\left(\frac{\partial V}{\partial u}\right) = m e^{mux} x^{n+1} (c - x)^\nu \text{ et } \left(\frac{\partial\partial V}{\partial u^2}\right) = mm e^{mux} x^{n+2} (c - x)^\nu.$$

Integrabilem ergo reddi oportet hanc formulam

$$e^{mux} x^n \partial x (c - x)^\nu (mmLxx + mMx + N),$$

cujus integrale ponatur $= e^{mux} x^{n+1} (c - x)^{\nu+1}$, cujus propterea differentiale illi formulae aequari debet: quod cum sit

$$e^{mux} x^n \partial x (c - x)^\nu [mux(c - x) + (n + 1)(c - x) - (\nu + 1)x],$$

erit

$$N = (n + 1)c,\; mM = mcu - (n + \nu + 2),\; mmL = -mu.$$

Statuatur nunc $x = a$, et formula

$$y = \int e^{mux} x^n \partial x (c - x)^\nu$$

erit integrale hujus aequationis

$$-\frac{u\partial\partial y}{m\partial u^2} + \frac{cu\partial y}{\partial u} - \frac{(n+\nu+2)\partial y}{m\partial u} + (n + 1)cy = e^{mau} a^{n+1} (c - a)^{\nu+1}.$$

Hic poni potest $m = 1$, ac sumto $c = a$, aequationis

$$u\partial\partial y - au\partial u\partial y + (n + \nu + 2)\partial u\partial y - (n + 1)ay\partial u^2 = 0,$$

integrale est $y = \int e^{ux} x^n \partial x (a - x)^\nu$, posito post integrationem $x = a$, dum sit $\nu + 1 > 0$, et $n + 1 > 0$, ut integrale evanescens reddi possit posito $x = 0$.

Corollarium 1.

1037. Si hic ponatur $y = e^{\int z \partial u}$, erit

$$u\partial z + uzz\partial u - auz\partial u + (n + \nu + 2)z\partial u - (n + 1)a\partial u = 0,$$

cujus integrale est

$$z = \frac{\partial y}{y \partial u} = \frac{\int e^{ux} x^{n+1} \partial x (a-x)^\nu}{\int e^{ux} x^n \partial x (a-x)^\nu}.$$

Illa aequatio autem posito $z = \frac{1}{2} a + v$, transmutatur in hanc

$$u \partial v + uvv \partial u + (n+\nu+2) v \partial u - \tfrac{1}{4} aau \partial u - \tfrac{1}{2}(n-\nu) a \partial u = 0,$$

quae ponendo $v = u^{-n-\nu-2} s$ abit in hanc

$$u^{-n-\nu-1} \partial s + u^{-2n-2\nu-3} ss \partial u - \tfrac{1}{4} aau \partial u - \tfrac{1}{2}(n-\nu) a \partial u = 0.$$

Corollarium 2.

1038. Sit porro

$$u^{-n-\nu-2} \partial u = \partial t \text{ seu } u^{-n-\nu-1} = -(n+\nu+1) t,$$

ut fiat

$$\partial s + ss \partial t - \tfrac{1}{4} aa u^{2n+2\nu+4} \partial t - \tfrac{1}{2}(n-\nu) u^{2n+2\nu+1} \partial t = 0,$$

quae ergo aequatio etiam construi potest. Vel sit

$$-(n+\nu+1) t = r, \text{ erit}$$

$$\partial s - \frac{ss \partial r}{n+\nu+1} + \frac{aar^{\frac{-2n-2\nu-4}{n+\nu+1}} \partial r}{4(n+\nu+1)} + \frac{(n-\nu) r^{\frac{-2n-2\nu-3}{n+\nu+1}} \partial r}{2(n+\nu+1)} = 0,$$

quae posito $s = -(n+\nu+1) q$ abit in

$$\partial q + qq \partial r - \frac{aar^{\frac{-2n-2\nu-4}{n+\nu+1}} \partial r - 2(n-\nu) r^{\frac{-2n-2\nu-3}{n+\nu+1}} \partial r}{4(n+\nu+1)^2} = 0,$$

hicque est

$$u = r^{\frac{-1}{n+\nu+1}} \text{ et } z = \tfrac{1}{2} a - (n+\nu+1) r^{\frac{n+\nu+2}{n+\nu+1}} q.$$

Scholion.

1039. Cum aequationis differentio - differentialis

$$\frac{\partial \partial y}{\partial u} - a \partial y + \frac{(n+\nu+2) \partial u}{u} - \frac{(n+1) a y \partial u}{u} = 0,$$

integrale sit $y = \int e^{ux} x^n \partial x (a - x)^\nu$, videamus, quomodo haec ipsa aequatio in alias formas transfundi possit. Sit primo $u = \alpha t^\lambda$, ideoque $\partial u = \alpha \lambda t^{\lambda - 1} \partial t$, unde fit

$$\frac{1}{\alpha\lambda}\partial \cdot \frac{\partial y}{t^{\lambda-1}\partial t} - a\partial y + \frac{(n+\nu+2)\partial y}{\alpha t^\lambda} - \frac{\lambda(n+1)ay\partial t}{t} = 0:$$

sumatur jam elementum ∂t constans, eritque

$$\frac{\partial\partial y}{\alpha\lambda t^{\lambda-1}\partial t} - \frac{(\lambda-1)\partial y}{\alpha\lambda t^\lambda} - a\partial y + \frac{(n+\nu+2)\partial y}{\alpha t^\lambda} - \frac{\lambda(n+1)ay\partial t}{t} = 0$$

seu

$$\partial\partial y - \alpha\lambda a t^{\lambda-1}\partial t\partial y + \frac{(\lambda n+\lambda\nu+\lambda+1)\partial t\partial y}{t} - \alpha\lambda\lambda(n+1)at^{\lambda-2}y\partial t^2 = 0,$$

cujus integrale est

$$y = \int e^{\alpha t^\lambda x} x^n \partial x (a - x)^\nu.$$

Ponatur porro

$$\frac{\partial y}{y} = \mathrm{P}\partial t + \frac{\partial z}{z}, \text{ ut sit } z = e^{-\int \mathrm{P}\partial t} y, \text{ erit}$$

$$\partial\partial z + 2\mathrm{P}\partial t\partial z - \alpha\lambda a t^{\lambda-1}\partial t\partial z + (\lambda n+\lambda\nu+\lambda+1)\frac{\partial t\partial z}{t} + z\partial t\partial \mathrm{P}$$
$$+ z\partial t^2[\mathrm{PP} - \alpha\lambda a t^{\lambda-1}\mathrm{P} + \frac{(\lambda n+\lambda\nu+\lambda+1)\mathrm{P}}{t} - \alpha\lambda\lambda(n+1)at^{\lambda-2}] = 0.$$

Ad terminos elemento ∂z affectos tollendos statuatur

$$\mathrm{P} = \tfrac{1}{2}\alpha\lambda a t^{\lambda-1} - \frac{(\lambda n+\lambda\nu+\lambda+1)}{2t},$$

fietque ac prodibit haec aequatio

$$\partial\partial z - z\partial t^2[\frac{(\lambda n+\lambda\nu+\lambda)^2-1}{4tt} + \tfrac{1}{2}\alpha\lambda\lambda(n-\nu)at^{\lambda-2} + \tfrac{1}{4}\alpha^2\lambda^2a^2t^{2\lambda-1}] = 0,$$

cujus propterea integrale est

$$z = e^{-\frac{1}{2}\alpha a t^\lambda} t^{\frac{\lambda n+\lambda\nu+\lambda+1}{2}} \int e^{\alpha t^\lambda x} x^n \partial x (a - x)^\nu.$$

Quodsi ergo sit $\nu = n$, $\lambda\lambda(2n+1)^2 - 1 = 0$, seu

$$\lambda = \frac{\pm 1}{2n+1}, \text{ et } \alpha = \pm\frac{2}{\lambda} = \pm 2(2n+1),$$

habebitur haec aequatio

$$\partial\partial z - aazt^{\frac{\pm 2}{2n+1} - 2}\partial t^2 = 0,$$

cujus integrale est

$$z = e^{\pm(2n+1)at^{\frac{\pm 1}{2n+1}}} t^{\mp\frac{1}{2}+\frac{1}{2}} \int e^{\mp 2(2n+1)t^{\frac{\pm 1}{2n+1}}x} x^n \partial x (a-x)^n.$$

Vel hujus aequationis

$$\partial\partial z - aat^{2\lambda-2} z\partial t^2 = 0$$

integrale est

$$z = e^{-\frac{a}{\lambda}t^\lambda} t \int e^{\frac{2x}{\lambda}t^\lambda} x^{\pm\frac{1}{2\lambda}-\frac{1}{2}}\partial x (a-x)^{\pm\frac{1}{2\lambda}-\frac{1}{2}}:$$

unde occasionem arripimus hujusmodi integrationes generalius investigandi.

Exemplum 4.

1040. *Si sint P et Q functiones quaecunque ipsius u, et capiatur*

$$y = P\int e^{Qx} x^{n-1}\partial x (a-x)^{\nu-1},$$

posito scilicet post integrationem $x = a$, erit hic valor ipsius y integrale cujuspiam aequationis differentio - differentialis

$$\frac{L\partial\partial y}{\partial u^2} + \frac{M\partial y}{\partial u} + Ny = 0$$

quae quaeritur.

Ad calculum contrahendum ponamus $\partial P = P'\partial u$ et $\partial P' = P''\partial u$, item $\partial Q = Q'\partial u$ et $\partial Q' = Q''\partial u$. Hinc erit

$$\frac{\partial y}{\partial u} = P'\int e^{Qx}x^{n-1}\partial x(a-x)^{\nu-1} + PQ'\int e^{Qx}x^n\partial x(a-x)^{\nu-1} \text{ et}$$

$$\frac{\partial\partial y}{\partial u^2} = P''\int e^{Qx}x^{n-1}\partial x(a-x)^{\nu-1} + 2P'Q'\int e^{Qx}x^n\partial x(a-x)^{\nu-1}$$
$$+ PQ''\int e^{Qx}x^n\partial x(a-x)^{\nu-1} + PQ'Q'\int e^{Qx}x^{n+1}\partial x(a-x)^{\nu-1}$$

unde colligitur

$$\frac{L\partial\partial y}{\partial u^2}+\frac{M\partial y}{\partial u}+Ny=$$
$$\int e^{Qx}x^{n-1}\partial x(a-x)^{\nu-1}\left\{\begin{array}{l}LP''+2LP'Q'x+LPQ''x+LPQ'Q'xx\\+MP'+MPQ'x+NP\end{array}\right\}$$

quod integrale statuatur $=e^{Qx}x^n(a-x)^\nu$, ita ut evanescat posito $x=a$, dum sit $\nu>0$, uti etiam evanescit casu $x=0$ si modo $\nu>0$. Cum igitur hujus formulae differentiale sit

$$e^{Qx}x^{n-1}\partial x(a-x)^{\nu-1}[Qx(a-x)+na-(n+\nu)x],$$

ejus comparatio cum forma inventa praebet

$LP''+MP'+NP=na$,
$2LP'Q'+LPQ'+MPQ'=aQ-(n+\nu)$, et
$LPQ'Q'=-Q$, ergo $L=\frac{-Q}{PQ'Q'}$, hinc
$M=\frac{aQ}{PQ'}-\frac{(n+\nu)}{PQ'}+\frac{2P'Q}{PPQ'Q'}+\frac{QQ''}{PQ'Q'Q'}$ et
$N=\frac{na}{P}+\frac{P''Q}{PPQ'Q'}-\frac{MP'}{P}$:

sicque aequatio differentio-differentialis erit cognita.

Corollarium 1.

1041. Si velimus ut sit $M=0$, erit

$$aQ-(n+\nu)+\frac{2P'Q}{PQ'}+\frac{QQ''}{Q'Q'}=0,$$

quae per $\frac{Q'\partial u}{Q}$ multiplicata abit in hanc

$$\frac{2\partial P}{P}+a\partial Q-\frac{(n+\nu)\partial Q}{Q}+\frac{\partial Q'}{Q'}=0,$$

cujus integrale est

$$\frac{e^{aQ}P^2Q'}{Q^{n+\nu}}=\text{Const. sive}$$

$$P=Ce^{-\frac{1}{2}aQ}Q^{\frac{n+\nu}{2}}\sqrt{\frac{\partial u}{\partial Q}}.$$

Corollarium 2.

1042. Sit $Q = 2\alpha u^\lambda$, erit $Q' = 2\alpha\lambda u^{\lambda-1}$, et

$$P = C e^{-\alpha a u^\lambda} u^{\frac{\lambda(n+\nu-1)+1}{2}},$$ hinc

$$L = -\frac{1}{2\alpha\lambda\lambda} e^{\alpha a u^\lambda} u^{\frac{-\lambda(n+\nu+1)+3}{2}},$$ et

$$N = \frac{na}{P} + \frac{Q\partial\partial P}{PP\partial Q^2},$$ at est

$$\frac{Q}{\partial Q^2} = \frac{u^{-\lambda+2}}{2\alpha\lambda\lambda\partial u^2},$$ et ob

$$\frac{\partial P}{P} = -\alpha\lambda a u^{\lambda-1}\partial u + \frac{\lambda(n+\nu-1)+1}{2} \cdot \frac{\partial u}{u},$$ erit

$$\frac{\partial\partial P}{P} = -\alpha\lambda(\lambda-1) a u^{\lambda-2}\partial u^2 - \frac{\lambda(n+\nu-1)-1}{2} \cdot \frac{\partial u^2}{uu} + \alpha\alpha\lambda\lambda a a u^{2\lambda-2}\partial u^2$$
$$- \alpha\lambda a[\lambda(n+\nu-1)+1] + \frac{[\lambda(n+\nu-1)+1]^2}{4}$$

seu

$$\frac{\partial\partial P}{P} = \alpha\alpha\lambda\lambda a a u^{2\lambda-2}\partial u^2 - \alpha\lambda\lambda(n+\nu) a u^{\lambda-2}\partial u^2 + \frac{\lambda\lambda(n+\nu-1)^2-1}{4} \cdot \frac{\partial u^2}{uu},$$

hinc

$$na + \frac{Q}{\partial Q^2} \cdot \frac{\partial\partial P}{P} = \tfrac{1}{2}\alpha a a u^\lambda + \tfrac{1}{2}(n-\nu)a + \frac{\lambda\lambda(n+\nu-1)^2-1}{8\alpha\lambda\lambda} u^{-\lambda},$$ et

$$N = e^{\alpha a u^\lambda} u^{\frac{-\lambda(n+\nu-1)-1}{2}} [\tfrac{1}{2}\alpha a a u^\lambda + \tfrac{1}{2}(n-\nu)a + \frac{\lambda\lambda(n+\nu-1)^2-1}{8\alpha\lambda\lambda} u^{-\lambda}].$$

Corollarium. 3.

1043. Hinc erit

$$\frac{N}{L} = -2\alpha\lambda\lambda u^{\lambda-2} [\tfrac{1}{2}\alpha a a u^\lambda + \tfrac{1}{2}(n-\nu)a + \frac{\lambda\lambda(n+\nu-1)^2-1}{8\alpha\lambda\lambda} u^{-\lambda}],$$

et hujus aequationis

$$\frac{\partial\partial y}{\partial u^2} = y\,[\alpha\alpha\lambda\lambda a a u^{2\lambda-2} + \alpha\lambda\lambda(n-\nu)\,a\,u^{\lambda-2} + \frac{\lambda\lambda(n+\nu-1)^2-1}{4uu}]$$

integrale est

$$y = e^{-\alpha a u^\lambda} u^{\frac{\lambda(n+\nu-1)+1}{2}} \int e^{2\alpha u^\lambda x} x^{n-1}\partial x\,(a-x)^{\nu-1}.$$

Ponamus $\alpha = \frac{1}{\lambda}$,

$$\lambda(n-\nu) = f, \text{ seu } \nu = n - \frac{f}{\lambda}, \text{ et } \frac{\lambda\lambda(n+\nu-1)^2 - 1}{4} = g,$$

unde fit

$$n = \frac{f+\lambda+\surd(1+4g)}{2\lambda} \text{ et } \nu = \frac{-f+\lambda+\surd(1+4g)}{2\lambda},$$

et hujus aequationis

$$\partial\partial y = y\,\partial u^2 (aa u^{2\lambda-2} + af u^{\lambda-2} + g u^{-2})$$

integrale est

$$y = e^{\frac{-a}{\lambda}u^\lambda} u^{\frac{\lambda(n+\nu-1)+1}{2}} \int e^{\frac{2x}{\lambda}u^\lambda} x^{n-1}\partial x(a-x)^{\nu-1}, \text{ seu}$$

$$y = e^{\frac{-a}{\lambda}u^\lambda} u^{\frac{1+\surd(1+4g)}{2}} \int e^{\frac{2x}{\lambda}u^\lambda} x^{\frac{f-\lambda+\surd(1+4g)}{2\lambda}} \partial x(a-x)^{\frac{-f-\lambda+\surd(1+4g)}{2\lambda}}.$$

Corollarium 4.

1044. Si ponamus $\alpha = \frac{-1}{\lambda}$,

$$\lambda(n-\nu) = -f, \text{ et } \frac{\lambda\lambda(n+\nu-1)^2-1}{4} = g \text{ erit}$$

$$n = \frac{-f+\lambda+\surd(1+4g)}{2\lambda} \text{ et } \nu = \frac{f+\lambda+\surd(1+4g)}{2\lambda},$$

unde hujus aequationis, quae cum praecedente convenit,

$$\partial\partial y = y\,\partial u^2 (aa u^{2\lambda-2} + af u^{\lambda-2} + g u^{-2})$$

integrale erit

$$y = e^{\frac{a}{\lambda}u^\lambda} u^{\frac{1+\surd(1+4g)}{2}} \int e^{\frac{-2x}{\lambda}u^\lambda} x^{\frac{-f-\lambda+\surd(1+4g)}{2\lambda}} \partial x(a-x)^{\frac{+f-\lambda+\surd(1+4g)}{2\lambda}}$$

ubi necesse est sit $n > 0$ et $\nu > 0$.

Exemplum 5.

1045. *Si ponamus* $y = \int \partial x\,(aa - xx)^{\nu-1} \cos.\alpha u^\lambda x$, *posito post integrationem* $x = a$, *ut* y *aequetur certae functioni ipsius* u, *invenire aequationem differentio-differentialem, cui ea satisfaciat.*

Cum sit

$$\frac{\partial y}{\partial u} = -\alpha\lambda u^{\lambda-1}\int x\partial x(aa-xx)^{\nu-1}\sin.\alpha u^{\lambda}x, \text{ et}$$

$$\frac{\partial\partial y}{\partial u^2} - \int x\partial x(aa-xx)^{\nu-1}[-\alpha\lambda(\lambda-1)u^{\lambda-2}\sin.\alpha u^{\lambda}x - \alpha\alpha\lambda\lambda u^{2\lambda-2}x\cos.\alpha u^{\lambda}x].$$

Hinc erit $\frac{L\partial\partial y}{\partial u^2} + \frac{M\partial y}{\partial u} + Ny =$

$$\int(aa-xx)^{\nu-1}\partial x\left\{\begin{array}{l} N\cos.\alpha u^{\lambda}x - \alpha\lambda M u^{\lambda-1}x\sin.\alpha u^{\lambda}x - \alpha\lambda(\lambda-1)Lu^{\lambda-2}x\sin.\alpha u^{\lambda}x \\ \qquad -\alpha\alpha\lambda\lambda Lu^{2\lambda-2}xx\cos\,\alpha u^{\lambda}x. \end{array}\right.$$

Fingatur integrale $= (aa-xx)^{\nu}\sin.\alpha u^{\lambda}x$, quod evanescit posito tam $x = 0$ quam $x = a$, repertiturque comparatione instituta

$$L = \frac{u^{-\lambda+2}}{\alpha\lambda\lambda},\ M = \frac{2\lambda\nu-\lambda+1}{\alpha\lambda\lambda}u^{-\lambda+1},\ N = \alpha aau^{\lambda}.$$

Quare hujus aequationis

$$\frac{\partial\partial y}{\partial u^2} + (2\lambda\nu-\lambda+1)\frac{\partial y}{u\partial u} + \alpha\alpha\lambda\lambda aau^{2\lambda-2}y = 0$$

integrale est

$$y = \int\partial x(aa-xx)^{\nu-1}\cos.\alpha u^{\lambda}x.$$

Corollarium 1.

1046. Si ergo sit $\nu = \frac{\lambda-1}{2\lambda}$ et $\alpha = \frac{1}{\lambda}$, hujus aequationis

$$\frac{\partial\partial y}{\partial u^2} + aau^{2\lambda-2}y = 0$$

integrale est

$$y = \int\partial x(aa-xx)^{\frac{-\lambda-1}{2\lambda}}\cos.\tfrac{1}{\lambda}u^{\lambda}x,$$

si quidem post integrationem statuatur $x = a$, integrali ita sumto ut evanescat posito $x = 0$.

Corollarium 2.

1047. Si igitur sit $\frac{-\lambda-1}{2\lambda} = i$ numero integro, seu $\lambda = \frac{-1}{2i+1}$, hujus aequationis

$$\partial\partial y + aau^{\frac{-4i-4}{2i+1}} y\partial u^2 = u$$

integrale est

$$y = \int \partial x (aa - xx)^i \cos. \tfrac{1}{\lambda} u^\lambda x$$

quod revera exhiberi potest. Prodeunt scilicet casus integrabiles supra indicati.

Scholion.

1048. Cum posuerimus $y = \int V \partial x$, existente V functione quacunque ipsarum u et x, quarum autem in hac integratione sola x ut variabilis tractatur, non opus est absolute integrale ita determinari, ut evanescat posito $x = 0$, sed sufficit ut certo quodam casu $x = b$ evanescat, quo facto si porro ponatur $x = a$, ut y aequetur functioni cuipiam ipsius u, quam per quadraturas assignare licet, quandoquidem hic integrationem formularum simplicium nobis concedi jure postulamus. Atque hic valor ipsius y per u expressus integrale exhibet cujusdam aequationis differentio-differentialis

$$L\partial\partial y + M\partial u\partial y + Ny\partial u^2 = U\partial u^2,$$

ubi autem necesse est, ut haec formula

$$\int \partial x \left[L\left(\frac{\partial\partial V}{\partial u^2}\right) + M\left(\frac{\partial V}{\partial u}\right) + NV\right]$$

integrari actu possit, quod integrale itidem ita est capiendum, ut evanescat posito $x = b$, tum vero posito $x = a$, id fiat $= U$.

Problema 131.

1049. Si fuerint P et Q functiones ipsius x, at K functio ipsius u, ac ponatur

$$y = \int P\partial x (\lambda + Q)^n,$$

integrali ita sumto ut evanescat casu $x = b$, tum vero statuatur $x = a$, ut pro y prodeat functio ipsius u, invenire aequationem

differentio - differentialem inter y et u, cui ille valor ipsius y satisfaciat.

Solutio.

Sit $\partial K = K' \partial u$ et $\partial K' = K'' \partial u$, et ob
$y = \int (K + Q)^n P \partial x$, erit $\frac{\partial y}{\partial u} = \int n K' (K + Q)^{n-1} P \partial x$,
ac denuo differentiando

$$\frac{\partial \partial y}{\partial u^2} = \int [n K'' (K + Q)^{n-1} + n(n-1) K' K' (K+Q)^{n-2}] P \partial x,$$

unde si L, M, N denotent functiones ipsius u, erit haec expressio

$$\frac{L \partial \partial y}{\partial u^2} + \frac{M \partial y}{\partial u} + N y = \int P \partial x (K + Q)^{n-2} \times$$

$$[N (K+Q)^2 + n M K' (K+Q) + n L K'' (K+Q) + n(n-1) L K' K']$$

$$= \int P \partial x (K+Q)^{n-2} \left\{ \begin{array}{l} N K K + n M K K' + n L K K'' + n(n-1) L K' K' \\ + 2 N K Q + n M K' Q + n L K'' Q + N Q Q \end{array} \right\},$$

quae cum debeat esse integrabilis, statuatur integrale

$$= R (K + Q)^{n-1} + \text{Const.}$$

ita ut evanescat posito ut ante $x = b$, ubi R sit functio ipsius x tantum. Cujus formae differentiale quia est

$$(K + Q)^{n-2} [K \partial R + Q \partial R + (n-1) R \partial Q],$$

oportet sit

$$[N K K + n M K K' + n L K K'' + n(n-1) L K' K'] P \partial x$$
$$+ (2 N K + n M K' + n L K'') P Q \partial x + N P Q Q \partial x$$
$$= K \partial R + Q \partial R + (n-1) R \partial Q.$$

Hic ergo duplicis generis termini adesse debent, alii ab u plane liberi, alii vero functione K affecti, quos deinceps seorsim aequari conveniet. Hunc in finem ponamus

$$N K K + n M K K' + n L K K'' + n(n-1) L K' K' = A + \alpha K,$$
$$2 N K + n M K' + n L K'' = B + \beta K, \text{ et}$$
$$N = C + \gamma K.$$

Ex binis prioribus elidendo M colligitur:

$$-NKK + n(n-1)LK'K' = A + \alpha K - BK - \beta KK,$$

unde ob $N = C + \gamma K$, concluditur

$$L = \frac{A + (\alpha - B)K - (\beta - C)KK + \gamma K^3}{n(n-1)K'K'},$$

hincque

$$M = \frac{B + \beta K - 2NK - nLK''}{nK'},$$

ita ut ex functione K litterae L, M et N, determinantur, dum A, α, B, β, C, γ constantes quascunque denotant. Nunc autem superest ut efficiatur

$$(A + \alpha K)P\partial x + (B + \beta K)PQ\partial x + (C + \gamma K)PQQ\partial x$$
$$= K\partial R + Q\partial R + (n-1)R\partial Q,$$

unde duplicis generis terminos seorsim aequando, fit

$$P\partial x(A + BQ + CQQ) = Q\partial R + (n-1)R\partial Q.$$
$$P\partial x(\alpha + \beta Q + \gamma QQ) = \partial R,$$

ideoque

$$\frac{A + BQ + CQQ}{\alpha + \beta Q + \gamma QQ} = Q + \frac{(n-1)R\partial Q}{\partial R}, \text{ seu}$$

$$\frac{(n-1)R\partial Q}{\partial R} = \frac{A + (B-\alpha)Q + (C-\beta)QQ - \gamma Q^3}{\alpha + \beta Q + \gamma QQ}, \text{ ergo}$$

$$\frac{\partial R}{R} = \frac{(n-1)\partial Q(\alpha + \beta Q + \gamma QQ)}{A + (B-\alpha)Q + (C-\beta)QQ - \gamma Q^3},$$

unde ex functione Q functio R definitur: tum vero erit

$$P\partial x = \frac{(n-1)R\partial Q}{A + (B-\alpha)Q + (C-\beta)QQ - \gamma Q^3}.$$

Abeat jam integrale illud $R(K+Q)^{n-1}$ + Const. in functionem U, posito $x = a$, ac valor initio assumtus

$$y = \int \frac{(n-1)R\partial Q(K+Q)^n}{A + (B-\alpha)Q + (C-\beta)QQ - \gamma Q^3}.$$

erit integrale hujus aequationis differentio-differentialis

$$L\partial\partial y + M\partial u\partial y + Ny\partial u^2 = U\partial u^2.$$

Corollarium 1.

1050. Cum pro Q functio quaecunque ipsius x accipi possit, nihil impedit, quo minus sumamus $Q = x$. Tum igitur quaeri oportet R ex hac aequatione

$$\frac{\partial R}{R} = \frac{(n-1)\partial x(\alpha + \beta x + \gamma x x)}{A + (B - \alpha)x + (C - \beta)xx - \gamma x^3},$$

eritque pro K functione quacunque ipsius u assumta

$$y = (n-1)\int \frac{R\partial x(K + x)^n}{A + (B - \alpha)x + (C - \beta)xx - \gamma x^3},$$

in quo, integrali ita sumto ut, posito $x = b$, evanescat, deinceps statui debet $x = a$.

Corollarium 2.

1051. Ex functione autem K aequatio differentio-differentialis ita formatur, ut sit

$$L = \frac{A - (B - \alpha)K + (C - \beta)KK + \gamma K^3}{n(n-1)\partial K^2}\partial u^2,$$

$$M = \frac{B - (2C - \beta)K - 2\gamma K^2}{n\partial K}\partial u - \frac{L\partial\partial K}{\partial u\,\partial K}, \text{ et } N = C + \gamma K.$$

Deinde in expressione $R(K + x)^{n-1} +$ Const. ita constituta, ut posito $x = b$ evanescat, ponatur $x = a$, et functio ipsius u inde resultans vocetur U, eritque aequatio differentio-differentialis

$$L\partial\partial y + M\partial u\partial y + Ny\partial u^2 = U\partial u^2.$$

Corollarium 3.

1052. Si expressio $R(K + x)^{n-1} +$ Const. ita sit comparata, ut utroque casu $x = b$ et $x = a$ evanescat, seu potius hi termini integrationis ita constituantur, ut hoc eveniat, formula pro y assumta satisfaciet huic aequationi

$$L\partial\partial y + M\partial u\partial y + Ny\partial u^2 = 0,$$

quae si deinceps in alias formas transmutetur, earum quoque integralia assignari poterunt.

Problema 132.

1053. Si fuerint P, Q functiones ipsius x, at K functio ipsius u, ac ponatur $y = \int e^{KQ} P \partial x$, integrali ita sumto ut evanescat casu $x = b$, tum vero ponatur $x = a$, et y aequabitur functioni ipsius u, quae satisfaciet cuipiam aequationi differentio-differentiali, quam invenire oportet.

Solutio.

Cum sit $y = \int e^{KQ} P \partial x$, erit

$$\frac{\partial y}{\partial u} = \int e^{KQ} K' P Q \partial x, \text{ et } \frac{\partial \partial y}{\partial u^2} = \int e^{NQ} P \partial x (K'' Q + K' K' Q Q),$$

unde fit

$$\frac{L \partial \partial y}{\partial u^2} + \frac{M \partial y}{\partial u} + N y = \int e^{KQ} P \partial x (N + M K' Q + L K'' Q + L K' K' Q Q),$$

cujus integrale statuatur $e^{KQ} R + \text{Const.}$ quae expressio evanescat posito $x = b$, fierique oportet

$$\partial R + K R \partial Q = P \partial x [N + (M K' + L K'') Q + L K' K' Q Q],$$

et ob rationes ante allegatas faciamus

$$L K' K' = A + \alpha K, \quad M K' + L K'' = B + \beta K, \quad N = C + \gamma K,$$

eritque

$$L = \frac{A + \alpha K}{K' K'} \text{ et } M = \frac{B + \beta K}{K'} - \frac{L K''}{K'},$$

atque obtinebimus has aequationes

$$\partial R = P \partial x (C + B Q + A Q Q),$$
$$R \partial Q = P \partial x (\gamma + \beta Q + \alpha Q Q),$$

unde colligitur

$$\frac{\partial R}{R} = \frac{\partial Q (C + B Q + A Q Q)}{\gamma + \beta Q + \alpha Q Q},$$

inventaque functione R, erit

$$P \partial x = \frac{R \partial Q}{\gamma + \beta Q + \alpha Q Q},$$

ita ut sit

$$y = \int e^{KQ} \frac{R \partial Q}{\gamma + \beta Q + \alpha QQ}.$$

Si jam expressio $e^{KQ} R$ + Const. posito $x = a$ abeat in functionem U, aequatio differentio-differentialis, cui hoc integrale convenit, erit

$$L \partial \partial y + M \partial u \partial y + N y \partial u^2 = U \partial u^2$$

Corollarium 1.

1054. Hic pro Q scribere licet x ut ante, unde fit

$$\frac{\partial R}{R} = \frac{\partial x (C + Bx + Axx)}{\gamma + \beta x + \alpha xx}, \text{ et } y = \int e^{Kx} \cdot \frac{R \partial x}{\gamma + \beta x + \alpha xx},$$

et U oritur ex forma $e^{Kx} R$ + Const. posito $x = a$. Valor autem ipsius R, pro ratione coëfficientium α, β, γ varias formas induere potest.

Corollarium 2.

1055. Pro K autem quaecunque functio ipsius u accipi potest, a cujus indole aequatio differentio-differentialis pendet. Erit autem

$$L = \frac{A + \alpha K}{\partial K^2} \partial u^2,$$

$$M = \frac{B + \beta K}{\partial K} \partial u - \frac{(A + \alpha K) \partial u \partial \partial K}{\partial K^3}, \text{ et } N = C + \gamma K,$$

unde aequatio differentio-differentialis est

$$\frac{(A + \alpha K) \partial \partial y}{\partial K^2} + \frac{(B + \beta K) \partial y}{\partial K} - \frac{(A + \alpha K) \partial \partial K \partial y}{\partial K^3} + (C + \gamma K) y = U.$$

Corollarium 3.

1056. Cum hic etiam u ex calculo excedat, perinde est cujusmodi functio ejus pro L assumatur, quin etiam sine detrimento amplitudinis poni potest $K = u$, dummodo ratio elementi, quod constans assumitur, habeatur.

Scholion 1.

1057. Si ergo sumatur $K = u$, atque elementum ∂u sumatur constans, ut fiat $\partial \partial K = u$, hinc ista aequatio construi potest

$$\frac{(A+\alpha u)\partial\partial y}{\partial u^2}+\frac{(B+\beta u)\partial y}{\partial u}+(C+\gamma u)y=U,$$

existente U ejusmodi functione ipsius u, quam descripsimus. Simili autem modo ex praecedente problemate construi potest haec aequatio

$$[A-(B-\alpha)u+(C-\beta)uu+\gamma u^3]\frac{\partial\partial y}{\partial u^2}+(n+1)[B-(2C-\beta)u-2\gamma uu]\frac{\partial y}{\partial u}$$
$$+n(n-1)(C+\gamma u)y=U,$$

quae aeque late patere est censenda, ac si functionem quamcunque ipsius u loco K scripsissemus. Hinc enim loco u scribendo functionem quamcunque ipsius t, ac ∂t pro constante sumendo, omnes illae formae derivari possunt. Ex quo haec aequatio multo latius patet illa, quam supra in genere per series infinitas resolvimus. Plerumque autem hae aequationes ita sunt comparatae, ut earum integratio aliis methodis expediri haud possit, quocirca haec methodus omnino digna videtur, ad quam ulterius excolendam Geometrae omnes vires intendant.

Scholion 2.

1058. In investigatione hujusmodi constructionum ita sum versatus, ut primo quasi per conjecturam formulam quandam differentialem $\int V\partial x=y$, in qua V erat certa functio ipsarum u et x, ubi autem u ut constans spectabatur, assumseram, indeque dato ipsi x valore tributo pertigerim ad aequationem differentio-differentialem inter u et y, cui formula illa assumta satisfaceret. Hic autem observandum est illam formulam integralem non prorsus ab arbitrio nostro pendere, sed certa quadam indole praeditam esse debere, ut evolutione facta res perducatur ad aequationem differentialem secundi gradus. Quamdiu autem hanc electionem soli conjecturae permittimus, perpaucae hujusmodi formulae menti se offerunt, quae ad scopum propositum perducunt: multoque minus sperare licet, ut hoc modo unquam ad datam aequationem differentio-differentialem perveniamus, casuique potissimum tribuendae videntur constructiones, quas hic tradidimus. Cum igitur longissime adhuc

simus remoti a solutione problematis, quo proposita quadam aequatione differentio-differentiali quaeritur formula illa ejus integrationem suppeditans, quod problema, an unquam solutionem sit nacturum, admodum incertum videtur; eo magis opera est adhibenda, ut saltem pro casibus particularibus investigationem formulae integrantis ex indole aequationis propositae derivare conemur, sicque quodam modo viam ad solutionem directam paremus. Ad hoc autem series infinitae, per quas hujusmodi aequationes supra resolvere docuimus, utiliter adhiberi possunt; unde in sequenti capite methodum exponam ex serie infinita solutionem cujuspiam aequationis differentio-differentialis continente, formulam illam integralem investigandi.

CAPUT XI.

DE

CONSTRUCTIONE AEQUATIONUM DIFFERENTIO - DIFFERENTIALIUM EX EARUM RESOLUTIONE PER SERIES INFINITAS PETITA.

Problema 133.

1059.

Proposita serie infinita

$$A + Bs + Cs^2 + \dots + Ms^{i-1} + Ns^i + \text{etc.}$$

in qua sit $B = \frac{0m+h}{1n+k}A$, $C = \frac{1m+h}{2n+k}B$, $D = \frac{2m+h}{3n+k}C$, et in genere $N = \frac{(i-1)m+h}{in+k}M$, ejus summam per formulam integralem exprimere.

Solutio.

Ponatur summa quaesita $= z$, ita ut sit

$$z = A + Bs + Cs^2 + Ds^3 + \dots + Ms^{i-1} + Ns^i + \text{etc.}$$

eritque differentiando

$$\frac{s\partial z}{\partial s} = 0A + 1Bs + 2Cs^2 + 3Ds^3 + \dots + (i-1)Ms^{i-1} + iNs^i + \text{etc.}$$

ex cujus combinatione cum praecedente oritur

$$\frac{ms\partial z}{\partial s} + hz = hA + (m+h)Bs + (2m+h)Cs^2 + \dots$$
$$+ [(i-1)m+h]Ms^{i-1} + (im+h)Ns^i + \text{etc.}$$

Deinde vero etiam simili modo est

$$\frac{ns\partial z}{\partial s} + kz = kA + (n+k)Bs + (2n+k)Cs^2 + \dots$$
$$+ [(i-1)n+k]Ms^{i-1} + (in+k)Ns^i + \text{etc.}$$

ergo ob

$$(n+k)\,B = h\,A,\ (2n+k)\,C = (m+h)\,B, \text{ etc. erit}$$

$$\frac{ns\partial z}{\partial s} + kz = kA + hAs + (m+h)Bs^2 + (2m+h)Cs^3 + \text{etc.}$$

unde manifesto conficitur

$$\frac{ns\partial z}{\partial s} + kz = kA + \frac{mss\partial z}{\partial s} + hsz, \text{ seu}$$

$$s\partial z(n - ms) + z\partial s(k - hs) = kA\partial s$$

hoc est

$$\partial z + \frac{z\partial s(k - bs)}{s(n - ms)} = \frac{kA\partial s}{s(n - ms)}.$$

Cum nunc sit

$$\frac{\partial s(k - bs)}{s(n - ms)} = \frac{k\partial s}{ns} + \frac{(mk - nb)\partial s}{n(n - ms)},$$

aequatio ista integrabilis fit multiplicata per

$$s^{\frac{k}{n}}(n - ms)^{\frac{nb - mk}{mn}}, \text{ proditque}$$

$$(n - ms)^{\frac{nb - mk}{ma}} s^{\frac{k}{n}} z = Ak\int s^{\frac{k}{n} - 1}\partial s(n - ms)^{\frac{nb - mk}{mn} - 1}$$

quod integrale ita capi oportet, ut posito $s = 0$ fiat $z = A$, quo observato habebimus

$$z = Aks^{-\frac{k}{n}}(n - ms)^{\frac{mk - nb}{mn}}\int s^{\frac{k}{n} - 1}\partial s(n - ms)^{\frac{nb - mk}{mn} - 1}.$$

Corollarium 1.

1060. Peculiari solutione eget casus $m = 0$, quo fit

$$\partial z + \frac{z\partial s(k - bs)}{ns} = \frac{Ak\partial s}{ns},$$

quae per $s^{\frac{k}{n}} e^{\frac{-bs}{n}}$ multiplicata praebet

$$e^{\frac{-bs}{n}} s^{\frac{k}{n}} z = \frac{Ak}{n}\int e^{\frac{-bs}{n}} s^{\frac{k}{n} - 1}\partial s$$

ideoque

$$z = \frac{Ak}{n} e^{\frac{bs}{n}} s^{\frac{-k}{n}} \int e^{\frac{-bs}{n}} \cdot s^{\frac{k}{n} - 1} \partial s,$$

integrali ita sumto, ut fiat $z = A$ posito $s = 0$.

Corollarium 2.

1061. Casus etiam $n = 0$ seorsim resolvi debet, aequatio enim

$$\partial z + z \partial s \left(\frac{bs - k}{mss}\right) = -\frac{Ak\partial s}{mss}$$

multiplicari debet per $s^{\frac{b}{m}} e^{\frac{k}{ms}}$, et invenitur integrale

$$e^{\frac{k}{ms}} s^{\frac{b}{m}} z = -\frac{Ak}{m} \int e^{\frac{k}{ms}} s^{\frac{b}{m} - 2} \partial s,$$

ideoque

$$z = -\frac{Ak}{m} e^{\frac{-k}{ms}} s^{\frac{-b}{m}} \int e^{\frac{k}{ms}} s^{\frac{b}{m} - 2} \partial s.$$

Corollarium 3.

1062. Si fuerit et $m = 0$ et $n = 0$, ob $N = \frac{b}{k} M$, series nostra erit geometrica, aequatio vero nostra erit

$$z \partial s (k - h s) = A k \partial s \text{ seu } z = \frac{Ak}{k - bs},$$

uti natura rei manifesto postulat.

Scholion.

1063. Imprimis hic casus notari meretur, quo est $k = 0$, et summa z sine signo integrali exprimi potest; erit namque

$$(n - m s)^{\frac{b}{m}} z = \text{Const.}$$

et quia si $s = 0$, fieri debet $z = A$, erit Const. $= A n^{\frac{b}{m}}$ ideoque

$$z = A n^{\frac{b}{m}} (n - m s)^{\frac{-b}{m}}, \text{ seu } z = A \left(\frac{n}{n - ms}\right)^{\frac{b}{m}},$$

vel etiam

$$z = A\left(1 - \frac{ms}{n}\right)^{\frac{-h}{m}}.$$

At vero integratio etiam succedit casu quo $k = n$, erit enim

$$(n - ms)^{\frac{h}{m} - 1} s z = A n \int \partial s\, (n - ms)^{\frac{h}{m} - 2},$$

quod integrale est $= \text{Const.} - \frac{An(n-ms)^{\frac{h}{m}-1}}{h-m}$, et quia posito $s = 0$, fit $z = A$, erit $0 = \text{Const.} - \frac{A}{h-m} \cdot n^{\frac{h}{m}}$, hincque

$$z = \frac{An}{(h-m)s}\left[\left(\frac{n}{n-ms}\right)^{\frac{h}{m}-1} - 1\right] = \frac{An}{(h-m)s}\left[\left(1 - \frac{ms}{n}\right)^{1-\frac{h}{m}} - 1\right].$$

Porro perspicuum est, integrationem expediri posse casu $k = 2n$, quo cum sit

$$(n - ms)^{\frac{h}{m} - 2} s s z = 2 A n \int s \partial s\, (n - ms)^{\frac{h}{m} - 3},$$

erit hoc integrale

$$= \text{Const.} - \frac{2Ans}{h-2m}(n-ms)^{\frac{h}{m}-2} + \frac{2An}{h-2m}\int \partial s\, (n-ms)^{\frac{h}{m}-2} \text{ vel}$$

$$= \text{Const.} - \frac{2Ans}{h-2m}(n-ms)^{\frac{h}{m}-2} - \frac{2An(n-ms)^{\frac{h}{m}-1}}{(h-m)(h-2m)};$$

ubi $\text{Const.} = \frac{2An^{\frac{h}{m}}}{(h-m)(h-2m)}$, ideoque

$$z = \frac{2Ann}{(h-m)(h-2m)ss}\left[\left(\frac{n}{n-ms}\right)^{\frac{h}{m}-2} - 1 - \frac{(h-2m)s}{n}\right];$$

similique modo etiam integratio casibus $k = 3n$, $k = 4n$ etc. absoetur.

Problema 134.

1064. Proposita hujusmodi serie infinita

$$A\mathfrak{A} + B\mathfrak{B}u + C\mathfrak{C}u^2 + D\mathfrak{D}u^3 + \ldots + M\mathfrak{M}u^{i-1} + \mathfrak{N}Nu^i + \text{etc.}$$

coëfficientium lege existente

$$\mathrm{B} = \frac{0m+b}{1n-k}\mathrm{A},\ \mathrm{C} = \frac{1m+b}{2n+k}\mathrm{B},\ \mathrm{D} = \frac{2m+b}{3n+k}\mathrm{C} \ldots \mathrm{N} = \frac{(i-1)m+b}{in+k}\mathrm{M},$$

$$\mathfrak{B} = \frac{0\mu+\eta}{1\nu+\theta}\mathfrak{A},\ \mathfrak{C} = \frac{1\mu+\eta}{2\nu+\theta}\mathfrak{B},\ \mathfrak{D} = \frac{2\mu+\eta}{3\nu+\theta}\mathfrak{C} \ldots \mathfrak{N} = \frac{(i-1)\mu+\eta}{i\nu+\theta}\mathfrak{M},$$

ejus summam per formulam integralem exprimere.

Solutio.

Posita summa

$$y = \mathrm{A}\mathfrak{A} + \mathrm{B}\mathfrak{B}u + \mathrm{C}\mathfrak{C}u^2 + \mathrm{D}\mathfrak{D}u^3 + \mathrm{E}\mathfrak{E}u^4 + \text{etc.}$$

consideretur series hoc modo formata

$$z = \mathrm{A} + \mathrm{B}ux + \mathrm{C}u^2x^2 + \mathrm{D}u^3x^3 + \mathrm{E}u^4x^4 + \text{etc.}$$

cujus summa posito $u^n x = s$ est, ut modo invenimus

$$z = \mathrm{A}ks^{-\frac{k}{n}}(n-ms)^{\frac{mk-nb}{mn}}\int s^{\frac{k}{n}-1}\partial s(n-ms)^{\frac{nb-mk}{mn}-1},$$

integrali ita determinato, ut posito $s = 0$ fiat $z = \mathrm{A}$.

Formetur hujusmodi formula integralis

$$\mathrm{V} = \int \mathrm{P}z\partial x = \int \mathrm{P}\partial x(\mathrm{A} + \mathrm{B}ux + \mathrm{C}u^2x^2 + \mathrm{D}u^3x^3 + \text{etc.})$$

in qua u spectetur ut constans, pro P autem ejusmodi functio ipsius x accipiatur, ut fiat

$$\int \mathrm{P}x\partial x = \frac{\mathfrak{B}}{\mathfrak{A}}\int \mathrm{P}\partial x,\ \int \mathrm{P}x^2\partial x = \frac{\mathfrak{C}}{\mathfrak{B}}\int \mathrm{P}x\partial x,\ \int \mathrm{P}x^3\partial x = \frac{\mathfrak{D}}{\mathfrak{C}}\int \mathrm{P}x^2\partial x, \text{ etc.}$$

postquam scilicet in his integralibus data lege sumtis variabili x datus quidem valor fuerit tributus. Cum igitur hinc sit

$$\int \mathrm{P}x\partial x = \frac{\mathfrak{B}}{\mathfrak{A}}\int \mathrm{P}\partial x,\ \int \mathrm{P}x^2\partial x = \frac{\mathfrak{C}}{\mathfrak{A}}\int \mathrm{P}\partial x,\ \int \mathrm{P}x^3\partial x = \frac{\mathfrak{D}}{\mathfrak{A}}\int \mathrm{P}\partial x, \text{ etc.}$$

erit

$$\mathrm{V} = \left(\mathrm{A} + \frac{\mathrm{B}\mathfrak{B}}{\mathfrak{A}}u + \frac{\mathrm{C}\mathfrak{C}}{\mathfrak{A}}u^2 + \frac{\mathrm{D}\mathfrak{D}}{\mathfrak{A}}u^3 + \text{etc.}\right)\int \mathrm{P}\partial x,$$

unde patet fore

$$y = \frac{\mathfrak{A}\mathrm{V}}{\int \mathrm{P}\partial x} = \frac{\mathfrak{A}\int \mathrm{P}z\partial x}{\int \mathrm{P}\partial x}.$$

Quare cum valor ipsius z sit cognitus, tantum superest, ut functio P ipsius x conditionibus memoratis praedita investigetur. In genere autem esse oportet

$$\int P x^i \partial x = \frac{\mathfrak{N}}{\mathfrak{M}} \int P x^{i-1} \partial x = \frac{(i-1)\mu + \eta}{i\nu + \theta} \int P x^{i-1} \partial x,$$

quae aequalitas cum sufficiat, ut tantum certo quodam casu, quo ipsi x datus tribuitur valor, subsistat, ponamus in genere esse

$$(i\nu + \theta)\int P x^i \partial x = [(i-1)\mu + \eta]\int P x^{i-1} \partial x + x^i Q,$$

ita ut pro terminis integralibus sit $Q = 0$. Differentiando ergo habebimus

$$(i\nu + \theta) P x^i \partial x = (i\mu - \mu + \eta) P x^{i-1} \partial x + x^i \partial Q + i x^{i-1} Q \partial x,$$

seu per x^{i-1} dividendo

$$(i\nu + \theta) P x \partial x = (i\mu - \mu + \eta) P \partial x + x \partial Q + i Q \partial x,$$

quae aequalitas cum pro omnibus valoribus ipsius i aeque subsistere debeat, hinc duas adipiscimur aequationes

$$\nu P x \partial x = \mu P \partial x + Q \partial x, \text{ et } \theta P x \partial x = (\eta - \mu) P \partial x + x \partial Q;$$

unde duplici modo colligimus

$$P \partial x = \frac{Q \partial x}{\nu x - \mu}, \text{ et } P \partial x = \frac{x \partial Q}{\theta x - (\eta - \mu)},$$

sicque alterum valorem per alterum dividendo

$$\frac{x \partial Q}{Q \partial x} = \frac{\theta x + \mu - \eta}{\nu x - \mu}, \text{ seu } \frac{\partial Q}{Q} = \frac{\partial x(\theta x + \mu - \eta)}{x(\nu x - \mu)},$$

quae evolvitur in

$$\frac{\partial Q}{Q} = \frac{\eta - \mu}{\mu x} \partial x + \frac{\mu\theta + \mu\nu - \eta\nu}{\mu(\nu x - \mu)} \partial x;$$

hinc integrando elicitur

$$Q = x^{\frac{\eta}{\mu} - 1} (\nu x - \mu)^{\frac{\theta\mu - \eta\nu}{\mu\nu} + 1} \text{ per Const. mult.}$$

seu

$$Q = - x^{\frac{\eta}{\mu} - 1} (\mu - \nu x)^{\frac{\theta\mu - \eta\nu}{\mu\nu} + 1},$$

unde fit

$$P\partial x = x^{\frac{\eta}{m}-1}\partial x(\mu-\nu x)^{\frac{\theta\mu-\eta\nu}{\mu\nu}}.$$

Cum igitur sit

$$(i\nu+\theta)\int Px^i\partial x = [(i-1)\mu+\eta]\int Px^{i-1}\partial x - x^{i+\frac{\eta}{\mu}-1}(\mu-\nu x)^{\frac{\theta\mu-\eta\nu}{\mu\nu}+1},$$

si haec integralia ita capiantur, ut evanescant posito $x=0$, tum vero statuatur $x=\frac{\mu}{\nu}$, fiet uti hypothesis nostra postulat

$$\int Px^i\partial x = \frac{(i-1)\mu+\eta}{i\nu+\theta}\int Px^{i-1}\partial x;$$

at vero in hunc finem necesse est, ut sit

$$i+\frac{\eta}{\mu}-1>0 \text{ et } \frac{\theta\mu-\eta\nu}{\mu\nu}+1>0.$$

Quae conditio si locum habeat, seriei propositae summa ita exprimetur, ut sit

$$y\int x^{\frac{\eta}{\mu}-1}\partial x(\mu-\nu x)^{\frac{\theta\mu-\eta\nu}{\mu\nu}} = \mathfrak{A}\int x^{\frac{\eta}{\mu}-1}z\partial x(\mu-\nu x)^{\frac{\theta\mu-\eta\nu}{\mu\nu}},$$

existente

$$z = Aks^{\frac{-k}{n}}(n-ms)^{\frac{mk-nb}{mn}}\int s^{\frac{k}{n}-1}\partial s(n-ms)^{\frac{nb-mk}{mn}-1},$$

integrali hoc ita sumto, ut fiat $z=A$ posito $s=0$. Hoc autem integrali invento, pro s scribatur ux, et hoc valore ipsius z in illa formula substituto, quantitatem u tanquam constantem tractari oportet, quoad illae integrationes lege praescripta fuerint absolutae, tum enim pro y prodibit functio ipsius u, summam seriei propositae exprimens.

Corollarium 1.

1065. Quia in geminatis coëfficientibus nostrae seriei similis lex progressionis assumitur, singulas series A, B, C, D etc. et $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, etc. inter se permutare licet, unde hac methodo duplex formula summam seriei exprimens obtinetur.

Corollarium 2.

1066. Etsi functio Q non in calculum ingreditur, eam tamen nosse oportet, quoniam ex ejus indole termini integrationis constitui debent, ita ut pro utroque fiat $Q = 0$. Hi scilicet termini sunt $x = 0$ et $x = \frac{\mu}{\nu}$, dum fuerit

$$i + \frac{\eta}{\mu} - 1 > 0 \text{ et } \frac{\theta\mu - \eta\nu}{\mu\nu} + 1 > 0,$$

ubi i est numerus integer positivus.

Corollarium 3.

1067. Pro functione Q casus quo vel $\mu = 0$ vel $\nu = 0$ seorsim sunt evolvendi. Priore quo $\mu = 0$, est

$$\frac{\partial Q}{Q} = \frac{\partial x(\theta x - \eta)}{\nu x x} = \frac{\theta \partial x}{\nu x} - \frac{\eta \partial x}{\nu x x}, \text{ unde fit}$$

$$Q = e^{\frac{\eta}{\nu x}} x^{\frac{\theta}{\nu}}.$$

Posteriori quo $\nu = 0$, est

$$\frac{\partial Q}{Q} = \frac{\partial x(\theta x + \mu - \eta)}{-\mu x} = -\frac{\theta \partial x}{\mu} + \frac{\eta - \mu}{\mu} \cdot \frac{\partial x}{x}$$

ideoque

$$Q = e^{\frac{-\theta x}{\mu}} x^{\frac{\eta}{\mu} - 1}.$$

Scholion.

1068. Constructiones hoc modo odornandae prorsus similes sunt iis, quas capite praecedente tradidimus, cum res etiam ad formulam integralem hujusmodi $\int V \partial x$ reducatur, in qua V est functio binarum variabilium u et x, quarum illa autem u in ipsa integratione constans reputatur, post integrationem vero ipsi x datus quidam valor assignatur. Verum tamen haec constructio ad casus in superiori methodo non contentos extenditur, quandoquidem fieri potest, ut quantitas z functiones maxime transcendentes involvat.

Vicissim autem vidimus, methodum praecedentem ad ejusmodi aequationes applicari posse, quae per series, quales hic tractamus, evolvi nequeant, unde in Analysin ex hoc fonte haud contemnenda incrementa hauriri posse videntur.

Problema 135.

1069. Proposita aequatione differentio-differentiali

$$xx(a+bx^n)\partial\partial y+(c+ex^n)\partial x\partial y+(f+gx^n)y\partial x^2=0,$$

valorem ipsius y per formulam integralem construere.

Solutio 1.

Hanc aequationem supra (967.) ita in seriem evolvimus, ut posito

$$y=x^\lambda(A+Bx^n+Cx^{2n}+Dx^{3n}+Ex^{4n}+\text{etc.}),$$

primo exponenti λ tribui debeat radix hujus aequationis

$$\lambda(\lambda-1)a+\lambda c+f=0,$$

tum vero posito brevitatis gratia

$$\lambda(\lambda-1)b+\lambda e+g=h \text{ fit}$$

$$B=\frac{-h}{n[na+(2\lambda-1)a+c]}A,$$

$$C=\frac{-nnb-(2\lambda-1)nb-ne-h}{2n[2na+(2\lambda-1)a+c]}B,$$

$$D=\frac{-4nnb-2(2\lambda-1)nb-2ne-h}{3n[3na+(2\lambda-1)a+c]}C,$$

ideoque, illius seriei positis binis terminis contiguis indefinite $Mx^{(i-1)n}+Nx^{in}$, generaliter

$$N=\frac{-(i-1)^2nnb-(2\lambda-1)(i-1)nb-(i-1)ne-h}{in[ina+(2\lambda-1)a+c]}M,$$

ubi cum denominator jam habeat factores, quales ante assumsimus, numeratorem quoque in factores resolvamus, quo nihilo aequali posito invenitur

$$(i-1)n=-\tfrac{1}{2}(2\lambda-1)-\frac{e}{2b}\pm\sqrt{\left[\tfrac{1}{4}(2\lambda-1)^2+\frac{(\lambda-1)e}{2b}+\frac{ee}{4bb}-\frac{b}{b}\right]},$$

seu

$$(i-1)n = -\tfrac{1}{2}(2\lambda-1) - \tfrac{e}{2b} \pm \tfrac{1}{2b}\sqrt{[(b-e)^2 - 4bg)}.$$

Ponatur brevitatis gratia

$$\sqrt{[(b-e)^2 - 4bg]} = q,$$

ut sit

$$(i-1)n = \frac{-(2\lambda-1)b - e \pm q}{2b},$$

et nostra relatio fiet

$$N = \frac{-[(i-1)nb + \frac{1}{2}(2\lambda-1)b + \frac{1}{2}e - \frac{1}{2}q][(i-1)nb + \frac{1}{2}(2\lambda-1)b + \frac{1}{2}e + \frac{1}{2}q]}{inb[ina + (2\lambda-1)a + c]}M.$$

Ponamus jam $x^n = u$, et seriem inventam ita repraesentemus

$$\frac{y}{x^\lambda} = A\mathfrak{A} + B\mathfrak{B}u + C\mathfrak{C}u^2 + \ldots\ldots + M\mathfrak{M}u^{i-1} + N\mathfrak{N}u^i + \text{etc.}$$

horumque duplicium coëfficientium lex ita se habebit

$$N = \frac{(i-1)nb + \frac{1}{2}(2\lambda-1)b + \frac{1}{2}e - \frac{1}{2}q}{-inb}M \text{ et}$$

$$\mathfrak{N} = \frac{(i-1)nb + \frac{1}{2}(2\lambda-1)b + \frac{1}{2}e + \frac{1}{2}q}{ina + (2\lambda-1)a + c}\mathfrak{M}.$$

Cum igitur haec series similis sit ei, quam ante construximus, comparationem instituamus, et habebimus

$$m = nb,\ h = \tfrac{1}{2}(2\lambda-1)b + \tfrac{1}{2}e - \tfrac{1}{2}q,$$

$$n = -nb, \text{ et } k = 0:$$

$$\mu = nb,\ \eta = \tfrac{1}{2}(2\lambda-1)b + \tfrac{1}{2}e + \tfrac{1}{2}q,$$

$$\nu = na, \text{ et } \theta = (2\lambda-1)a + c.$$

Primum ergo quaeramus quantitatem z, et ne littera x ambiguitatem creet, loco litterae x in praecedente problemate usurpatae utamur littera t, sitque $ut = s$, et quia est $k = 0$, erit per §. 1063.

$$z = A(1+s)^{\frac{-(2\lambda-1)b-e+q}{2nb}} = A(1+ut)^{\frac{-(2\lambda-1)b-e+q}{2nb}}.$$

Hoc valore invento tractetur in sequentibus integrationibus quantitas u ut constans, et cum quod supra erat y hic sit $\frac{y}{x^\lambda}$, et t quod supra erat x, erit

$$\frac{y}{x^\lambda}\int t^{\frac{\eta}{\mu}-1}\partial t(\mu-\nu t)^{\frac{\theta\mu-\eta\nu}{\mu\nu}}=\mathfrak{A}\int t^{\frac{\eta}{\mu}-1}z\partial t(\mu-\nu t)^{\frac{\theta\mu-\eta\nu}{\mu\nu}},$$

ubi cum sit $u=x^n$, hic valor statim pro u scribi potest, ut sit

$$z=A(1+x^n t)^{\frac{-(2\lambda-1)b-e+q}{2nb}},$$

et in his integrationibus littera x ut constans spectari debet. Quodsi autem fuerit

$$i+\frac{\eta}{\mu}-1>0 \text{ et } \frac{\theta\mu-\eta\nu}{\mu\nu}+1>0,$$

haec integralia ita capi debent, ut evanescant posito $t=0$, quo facto ipsi t tribui debet valor $t=\frac{\mu}{\nu}=\frac{b}{a}$. Cum unitas sit minimus valor ipsius i, sufficit ut sit

$$\frac{(2\lambda-1)b+e-q}{2nb}>0, \text{ tum vero } \frac{(2\lambda+1)ab+2bc-ae-aq}{2nab}+1>0.$$

Nunc vero cum

$$\int t^{\frac{\eta}{\mu}-1}\partial t(\mu-\nu t)^{\frac{\theta\mu-\eta\nu}{\mu\nu}},$$

fiat quantitas constans, pro y autem ejus multiplum quodvis nostrae aequationi aeque satisfaciat, ejus integrale ita exprimetur

$$y=Cx^\lambda\int t^{\frac{\eta}{\mu}-1}z\partial t(\mu-\nu t)^{\frac{\theta\mu-\eta\nu}{\mu\nu}},$$

existente

$$z=(1+x^n t)^{\frac{-(2\lambda-1)b-e+q}{2nb}}.$$

Solutio II.

1071. Si coëfficientes geminatos inter se permutemus, ut sit

$$m = n b, \quad h = \tfrac{1}{2}(2\lambda - 1) b + \tfrac{1}{2} e + \tfrac{1}{2} q,$$
$$n = n a, \quad k = (2\lambda - 1) a + c,$$
$$\mu = n b, \quad \eta = \tfrac{1}{2}(2\lambda - 1) b + \tfrac{1}{2} e - \tfrac{1}{2} q,$$
$$\nu = - n b, \ \theta = 0,$$

sumaturque λ ex aequatione

$$\lambda(\lambda - 1) a + \lambda c + f = 0.$$

Primo ponatur $x^n t = s$, et quaeratur z ut sit

$$z = A k s^{\frac{-k}{n}} (n - ms)^{\frac{mk - nb}{mn}} \int s^{\frac{k}{n} - 1} \partial s \, (n - ms)^{\frac{nb - mk}{mn} - 1},$$

integrali ita definito, ut posito $s = 0$ fiat $z = A$, qui quidem valor A est arbitrarius, tum spectata x ut constante erit

$$y = C x^\lambda \int t^{\frac{\eta}{\mu} - 1} z \, \partial t \, (\mu - \nu t)^{\frac{\theta\mu - \eta\nu}{\mu\nu}},$$

integrali ita sumto ut evanescat posito $t = 0$; tum vero facto $t = \frac{\mu}{\nu}$, si modo fuerit $\frac{\eta}{\mu} > 0$ et $1 - \frac{\eta}{\mu} > 0$, ob $\theta = 0$; ubi notetur z per hanc aequationem differentialem definiri

$$\frac{\partial z}{\partial s} = \frac{A k - z(k - bs)}{s(n - ms)}.$$

Solutio III.

1072. Per seriem descendentem aequationem propositam ita resolvimus, ut posito

$$y = x^\lambda (A + B x^{-n} + C x^{-2n} + D x^{-3n} + \text{etc.})$$

exponens λ definiri debeat ex hac aequatione

$$\lambda(\lambda - 1) b + \lambda e + g = 0,$$

tum vero posito $\lambda(\lambda - 1) a + \lambda c + f = h$, sit

$$B = \frac{-b}{n[nb - (2\lambda - 1) b - e]} A,$$
$$C = \frac{-nna + (2\lambda - 1) na - nc - b}{2n[2nb - (2\lambda - 1) b - e]} B,$$

**

et generatim

$$N = \frac{-(i-1)^2 nna + (2\lambda - 1)(i-1)na + (i-1)nc - b}{in[inb - (2\lambda - 1)b - e]} M,$$

quae aequalitas posito $\sqrt{[(a-c)^2 - 4af]} = p$, ita per factores exhibetur

$$N = \frac{-[(i-1)na - \frac{1}{2}(2\lambda - 1)a - \frac{1}{2}c - \frac{1}{2}p][(i-1)na - \frac{1}{2}(2\lambda - 1)a - \frac{1}{2}c + \frac{1}{2}p]}{ina[inb - (2\lambda - 1)b - e]} M.$$

Quodsi jam ponamus $x^{-n} = u$, et talem seriem constituamus

$$\frac{y}{x^\lambda} = A\mathfrak{A} + B\mathfrak{B}u + C\mathfrak{C}u^2 + \ldots\ldots + M\mathfrak{M}u^{i-1} + N\mathfrak{N}u^i + \text{etc.}$$

erit

$$N = \frac{(i-1)na - \frac{1}{2}(2\lambda - 1)a - \frac{1}{2}c - \frac{1}{2}p}{-ina} M, \text{ et}$$

$$\mathfrak{N} = \frac{(i-1)na - \frac{1}{2}(2\lambda - 1)a - \frac{1}{2}c + \frac{1}{2}p}{inb - (2\lambda - 1)b - e} \mathfrak{M},$$

et habebimus comparatione instituta cum constructione generali

$m = na$, $h = -\frac{1}{2}(2\lambda - 1)a - \frac{1}{2}c - \frac{1}{2}p$,
$n = -na$, et $k = 0$,
$\mu = na$, $\eta = -\frac{1}{2}(2\lambda - 1)a - \frac{1}{2}c + \frac{1}{2}p$,
$\nu = nb$, et $\theta = -(2\lambda - 1)b - e$.

Hinc posito $s = ut = x^{-n}t$, erit

$$z = A(1+s)^{\frac{-b}{m}} = A(1 + x^{-n}t)^{\frac{-b}{m}},$$

quo valore invento, si jam sola quantitas t pro variabili habeatur, orietur haec constructio

$$y = Cx^\lambda \int t^{\frac{\eta}{\mu} - 1} z\, \partial t\, (\mu - \nu t)^{\frac{\theta\mu - \eta\nu}{\mu\nu}},$$

ubi termini integrationis ita sunt constituendi, ut utroque fiat

$$t^{\frac{\eta}{\mu}} (\mu - \nu t)^{\frac{\theta\mu - \eta\nu}{\mu\nu} + 1} = 0.$$

Solutio. IV.

1703. Hic etiam coëfficientes geminati permutari possunt, ut sit

$$m = na, \; h = -\tfrac{1}{2}(2\lambda - 1)a - \tfrac{1}{2}c + \tfrac{1}{2}p,$$
$$n = nb, \; k = -(2\lambda - 1)b - e,$$
$$\mu = na, \; \eta = -\tfrac{1}{2}(2\lambda - 1)a - \tfrac{1}{2}c - \tfrac{1}{2}p,$$
$$\nu = -na, \text{ et } \theta = 0,$$

sumtoque ut ante λ ex aequatione

$$\lambda(\lambda - 1)b + \lambda e + g = 0,$$

et posito $\sqrt{[(a-c)^2 - 4af)} = p$, sit $s = x^{-n}t$, et quaeratur z ex hac aequatione

$$\frac{\partial z}{\partial s} = \frac{Ak - z(k - bs)}{s(n - ms)},$$

ita ut posito $s = 0$ fiat $z = A$, unde fit

$$z = Aks^{\frac{-k}{n}}(n - ms)^{\frac{mk - nb}{mn}} \int s^{\frac{k}{n} - 1} \partial s (n - ms)^{\frac{nb - mk}{mn} - 1},$$

tum vero spectata x ut constante, erit

$$y = C x^\lambda \int t^{\frac{\eta}{\mu} - 1} z \partial t (\mu - \nu t)^{\frac{\theta\mu - \eta\nu}{\mu\nu}},$$

ubi bini termini integrationis ita sumi debent, ut utroque fiat

$$t^{\frac{\eta}{\mu}}(\mu - \nu t)^{\frac{\theta\mu - \eta\nu}{\mu\nu} + 1} = 0.$$

Scholion.

1074. Singulae hae constructiones plerumque pluribus modis exhiberi possunt, cum non solum λ duplicem valorem habere queat, sed etiam formulae radicales p et q signo ambiguo gaudeant. At hae constructiones alio modo negotium conficiunt atque praecedentes, quod quo clarius appareat, consideremus aequationem

$$xx(1-xx)\partial\partial y - x(1+xx)\partial x\partial y + xxy\partial x^2 = 0,$$

ita ut sit $a=1$, $b=-1$, $c=-1$, $e=-1$, $f=0$ et $g=1$, atque $n=2$; unde pro duabus prioribus constructionibus habetur $\lambda(\lambda-1)-\lambda=0$, ergo vel $\lambda=0$ vel $\lambda=2$, tum vero $q=\pm 2$. Constructio ergo prima dat $\lambda=0$, $m=-2$, $h=\mp 1$, $n=2$, $k=0$, $\mu=-2$, $\eta=\pm 1$, $\nu=2$, $\theta=-2$; unde colligitur

$$z=(1+xxt)^{\mp\frac{1}{2}}, \text{ et } y=C\int t^{\mp\frac{1}{2}-1}z\partial t(1+t)^{\pm\frac{1}{2}-1}.$$

Valeant signa inferiora, ut sit

$$z=\sqrt{(1+xxt)} \text{ et } y=C\int\frac{z\partial t}{(1+t)\sqrt{t(1+t)}}.$$

Quae formulae quomodo satisfaciant, videamus. Sumta nempe sola x variabili, fit

$$\frac{\partial z}{\partial x}=\frac{xt}{\sqrt{(1+xxt)}} \text{ et } \frac{\partial\partial z}{\partial x^2}=\frac{t}{(1+xxt)^{\frac{3}{2}}},$$

hinc

$$\frac{\partial y}{\partial x}=C\int\frac{xt\partial t}{t^{\frac{1}{2}}(1+t)^{\frac{3}{2}}(1+xxt)^{\frac{1}{2}}} \text{ et } \frac{\partial\partial y}{\partial x}=C\int\frac{t\partial t}{t^{\frac{1}{2}}(1+t)^{\frac{3}{2}}(1+xxt)^{\frac{3}{2}}}$$

unde conficitur

$$xx(1-xx)\frac{\partial\partial y}{\partial x^2}-x(1+xx)\frac{\partial y}{\partial x}+xxy=C\int\frac{xx\partial t(1-xxtt)}{t^{\frac{1}{2}}(1+t)^{\frac{3}{2}}(1+xxt)^{\frac{3}{2}}},$$

cujus formulae integrale est $\frac{2Cxx\sqrt{t}}{\sqrt{(1+t)(1+xxt)}}$; quod cum evanescat tam casu $t=0$ quam casu $t=\infty$, constructio nostrae aequationis

$$y=C\int\frac{z\partial t}{(1+t)\sqrt{t(1+t)}}=C\int\frac{\partial t\sqrt{(1+xxt)}}{(1+t)\sqrt{t(1+t)}},$$

ita confici debet; sumto x pro constante, integratio ita instituatur, ut integrale evanescat posito $t=0$, quo facto statuatur $t=\infty$, et functio ipsius x, quae pro y prodibit, satisfaciet aequationi propositae.

Sin autem secundam constructionem eligamus, sumta $\lambda = 0$, erit $m = -2$, $h = \pm 1$, $n = 2$, $k = -2$, $\mu = -2$, $\eta = \mp 1$, $\nu = 2$, $\theta = 0$, atque z ita definiri debet ex hac aequatione

$$\frac{\partial z}{\partial s} = \frac{-2A + z(2 \pm s)}{s(2 + 2s)},$$

existente $s = xxt$, ut posito $s = 0$ fiat $z = A$.

Valeat signum superius, et cum sit

$$\partial z - \frac{z \partial s (2 + s)}{2s(1 + s)} = \frac{-A \partial s}{s(1 + s)}:$$

multiplicetur per $\frac{\sqrt{(1+s)}}{s}$, eritque integrale

$$\frac{z\sqrt{(1+s)}}{s} = \text{Const.} - A \int \frac{\partial s}{ss\sqrt{(1+s)}}, \text{ seu}$$

$$z = A - \frac{As}{\sqrt{(1+s)}} l \frac{1 + \sqrt{(1+s)}}{\sqrt{s}} + \frac{Bs}{\sqrt{(1+s)}},$$

quae posito $s = 0$ dat $z = A$, quicquid sit B. Deinde est

$$y = C \int t^{\frac{1}{2} - 1} z \partial t (1 + t)^{-\frac{1}{2}} \text{ seu } y = C \int \frac{z \partial t}{\sqrt{t(1+t)}},$$

qui valor quomodo satisfaciat haud facile ostendi potest; hocque magis ista methodus excoli meretur.

Exemplum.

1075. *Constructiones aequationis differentio-differentialis*

$$xx(1 - xx)\partial\partial y - x(1 + xx)\partial x \partial y + xxy\partial x^2 = 0$$

ex praecedente problemate oriundas exhibere.

Ob $n = 2$, $a = 1$, $b = -1$, $c = -1$, $e = -1$, $f = 0$, et $g = 1$, pro prima constructione habemus vel $\lambda = 0$ vel $\lambda = 2$, unde obtinemus.

1.) Si $\lambda = 0$, ut modo invenimus

$m = -2$, $h = \mp 1$, $n = 2$, $k = 0$, $\mu = -2$, $\eta = \pm 1$, $\nu = 2$, $\theta = -2$, unde fit

$$z=(1\mp xxt)^{\mp\frac{1}{2}} \text{ et } y=C\int t^{\mp\frac{1}{2}-1}\, z\partial t(1+t)^{\pm\frac{1}{2}-1},$$

sicque duplex oritur constructio

altera $z=\sqrt{(1+xxt)}$ et $y=C\int\frac{z\partial t}{(1+t)\sqrt{t(1+t)}}$,

altera $z=\frac{1}{\sqrt{(1+xxt)}}$ et $y=C\int\frac{z\partial t}{t\sqrt{t(1+t)}}$.

2.) Si $\lambda=2$, erit ob $q=\pm 2$,

$m=-2$, $h=-2\mp 1$, $n=2$, $k=0$, $\mu=-2$,
$\eta=-2\pm 1$, $\nu=2$, $\theta=2$,

unde fit

$$z=(1+xxt)^{\frac{-2\mp 1}{2}} \text{ et } y=Cx^2\int t^{\mp 1\mp\frac{1}{2}-1}\, z\partial t(1+t)^{\pm\frac{1}{2}},$$

sicque duplex habetur constructio

altera $z=(1+xxt)^{-\frac{3}{2}}$ et $y=Cx^2\int\frac{z\partial t\sqrt{(1+t)}}{\sqrt{t}}$,

altera $z=(1+xxt)^{-\frac{1}{2}}$ et $y=Cx^2\int\frac{z\partial t\sqrt{t}}{\sqrt{(1+t)}}$,

Pro secunda constructione generali habemus:

3.) Si $\lambda=0$, permutando illos indices,

$m=-2$, $h=\pm 1$, $n=2$, $k=-2$; $\mu=-2$, $\eta=\mp 1$,
$\nu=2$, $\theta=0$,

unde posito $xxt=s$, primo quaeratur z ex hac aequatione $\frac{\partial z}{\partial s}=\frac{-2A+z(2\pm s)}{2s(1+s)}$, ut posito $s=0$ fiat $z=A$; tum vero erit

$$y=C\int t^{\pm\frac{1}{2}-1}\, z\partial t(1+t)^{\mp\frac{1}{2}}.$$

Hinc ergo nascitur duplex constructio

altera $\frac{\partial z}{\partial s}=\frac{-2A+z(2+s)}{2s(1+s)}$ et $y=C\int\frac{z\partial t}{\sqrt{t(1+t)}}$,

altera $\frac{\partial z}{\partial s}=\frac{-2A+z(2-s)}{2s(1+s)}$ et $y=C\int\frac{z\partial t\sqrt{(1+t)}}{t\sqrt{t}}$,

4.) Si $\lambda=2$ habebimus

$m=-2$, $h=-2\pm 1$, $n=2$, $k=2$, $\mu=-2$,
$\eta=-2\mp 1$, $\nu=2$, $\theta=0$,

positoque $xxt = s$, ut ante, quaeratur z ex aequatione

$$\frac{\partial z}{\partial s} = \frac{2A - z[2 + (2 \mp 1)s]}{2s(1+s)}, \text{ eritque}$$

$$y = C x^2 \int t^{\pm \frac{1}{2}} z \partial t (1+t)^{\frac{-2 \mp 1}{2}};$$

deoque etiam duplex constructio elicitur

altera $\frac{\partial z}{\partial s} = \frac{2A - z(2+s)}{2s(1+s)}$ et $y = C x^2 \int \frac{z \partial t \sqrt{t}}{(1+t)\sqrt{(1+t)}}$

altera $\frac{\partial z}{\partial s} = \frac{2A - z(2+3s)}{2s(1+s)}$ et $y = C x^2 \int \frac{z \partial t}{\sqrt{t(1+t)}}$.

Ex solutione tertia colligimus primo

$$-\lambda(\lambda - 1) - \lambda + 1 = 0 \text{ seu } \lambda\lambda = 1,$$

ideoque

$$\lambda = \pm 1 \text{ et } p = \sqrt{4} = \pm 2; \text{ quare}$$

5.) si capiatur $\lambda = +1$, erit

$$m = 2,\ h = \mp 1,\ n = -2,\ k = 0,\ \mu = 2,\ \eta = \pm 1,$$
$$\nu = -2,\ \theta = 2,$$

hincque

$$z = \left(1 + \frac{t}{xx}\right)^{\pm \frac{1}{2}} \text{et } y = C x \int t^{\pm \frac{1}{2} - 1} z \partial t (1+t)^{-1 \mp \frac{1}{2}},$$

ita ut iterum duplex habeatur constructio

altera $z = \frac{1}{x}\sqrt{(xx+t)}$ et $y = C x \int \frac{z \partial t}{(1+t)\sqrt{t(1+t)}}$

altera $z = \frac{x}{\sqrt{(xx+t)}}$ et $y = C x \int \frac{z \partial t}{t\sqrt{t(1+t)}}$.

6.) Si capiatur $\lambda = -1$ erit

$$m = 2,\ h = 2 \mp 1,\ n = -2,\ k = 0,\ \mu = 2,\ \eta = 2 \pm 1,$$
$$\nu = -2,\ \theta = -2,$$

hincque

$$z = \left(1 + \frac{t}{xx}\right)^{-1 \pm \frac{1}{2}} \text{ et } y = \frac{C}{x} \int t^{\pm \frac{1}{2}} z \partial t (1+t)^{\mp \frac{1}{2}},$$

unde binae constructiones fluunt

$$\text{altera}\quad z = \frac{x}{\sqrt{(xx+t)}} \quad \text{et} \quad y = \frac{C}{x}\int \frac{z\,\partial t.\sqrt{t}}{\sqrt{(1+t)}}$$

$$\text{altera}\quad z = \frac{x^3}{(xx+t)^{\frac{3}{2}}} \quad \text{et} \quad y = \frac{C}{x}\int \frac{z\,\partial t\sqrt{(1+t)}}{\sqrt{t}}.$$

Ex solutione quarta denique concludimus

7.) Si $\lambda = +1$,

$m=2$, $h=\pm 1$, $n=-2$, $k=2$, $\mu=2$, $\eta=\mp 1$, $\nu=-2$, $\theta=0$.

Posito nunc $s=\frac{t}{xx}$, quaeratur z ex hac aequatione

$$\frac{\partial z}{\partial s} = \frac{-2A+z(2\mp s)}{2s(1+s)},$$

ut posito $s=0$ fiat $z=A$, tumque erit

$$y = Cx\int t^{\mp\frac{1}{2}-1}\, z\,\partial t\,(1+t)^{\pm\frac{1}{2}},$$

unde duplex constructio

$$\text{altera}\quad \frac{\partial z}{\partial s} = \frac{-2A+z(2-s)}{2s(1+s)} \quad \text{et} \quad y = Cx\int\frac{z\,\partial t\sqrt{(1+t)}}{t\sqrt{t}}$$

$$\text{altera}\quad \frac{\partial z}{\partial s} = \frac{-2A+z(2+s)}{2s(1+s)} \quad \text{et} \quad y = Cx\int\frac{z\,\partial t}{\sqrt{t(1+t)}}.$$

8.) Si $\lambda = -1$, habebitur

$m=2$, $h=2\pm 1$, $n=-2$, $k=-2$, $\mu=2$, $\eta=2\mp 1$, $\nu=-2$; $\theta=0$,

at posito $\frac{t}{xx}=s$, quaeri debet z ex hac aequatione

$$\frac{\partial z}{\partial s} = \frac{\pm 2A - z\,[2+(2\pm 1)s]}{2s(1+s)},$$

ut posito $s=0$ fiat $z=A$, quo facto erit

$$y = \frac{C}{x}\int t^{\mp\frac{1}{2}}\, z\,\partial t\,(1+t)^{-1\pm\frac{1}{2}};$$

sicque duplex oritur constructio

altera $\frac{\partial z}{\partial s} = \frac{2A - z(2+3s)}{2s(1+s)}$ et $y = \frac{C}{x}\int\frac{z\partial t}{\sqrt{t(1+t)}}$

altera $\frac{\partial z}{\partial s} = \frac{2A - z(2+s)}{2s(1+s)}$ et $y = \frac{C}{x}\int\frac{z\partial t\sqrt{t}}{(1+t)\sqrt{(1+t)}}$.

Omnino ergo sedecim constructiones sumus consecuti.

Scholion.

1076. Periculum faciamus ostendendi, quomodo hae constructiones, quae magis arduae videntur, aequationi propositae satisfaciant; atque in hunc finem eligamus constructionem posteriorem No. 4. quae habet

$$\partial z + \frac{z\partial s(2+3s)}{2s(1+s)} = \frac{A\partial s}{s(1+s)},$$

haee per $s\sqrt{(1+s)}$ multiplicata praebet integrale

$$sz\sqrt{(1+s)} = A\int\frac{\partial s}{\sqrt{(1+s)}} = 2A\sqrt{(1+s)} + B, \text{ seu}$$
$$z = \frac{2A}{s} + \frac{B}{s\sqrt{(1+s)}}.$$

Jam ut posito $s = 0$ fiat $z = A$, debet esse $B = -2A$, ut sit

$$z = \frac{2A[\sqrt{(1+s)} - 1]}{s\sqrt{(1+s)}} = \frac{2A}{txx} - \frac{2A}{txx\sqrt{(1+txx)}}.$$

Hinc fit

$$\left(\frac{\partial z}{\partial x}\right) = \frac{-4A}{tx^3} + \frac{2A(2+3txx)}{tx^3(1+txx)^{\frac{3}{2}}} \text{ et}$$

$$\left(\frac{\partial\partial z}{\partial x^2}\right) = \frac{12A}{tx^4} - \frac{6A(2+5txx+4ttx^4)}{tx^4(1+txx)^{\frac{5}{2}}}.$$

Cum nunc sit $y = C\int\frac{xxz\partial t}{\sqrt{t(1+t)}}$, erit

$$\left(\frac{\partial y}{\partial x}\right) = 2C\int\frac{xz\partial t}{\sqrt{t(1+t)}} + C\int\frac{xx\partial t}{\sqrt{t(1+t)}}\cdot\left(\frac{\partial z}{\partial x}\right) \text{ et}$$

$$\left(\frac{\partial\partial y}{\partial x^2}\right) = 2C\int\frac{z\partial t}{\sqrt{t(1+t)}} + 4C\int\frac{x\partial t}{\sqrt{t(1+t)}}\left(\frac{\partial z}{\partial x}\right) + C\int\frac{xx\partial t}{\sqrt{t(1+t)}}\left(\frac{\partial\partial z}{\partial x^2}\right);$$

hineque

$$x(1-xx)\frac{\partial\partial y}{\partial x^2}-x(1+xx)\frac{\partial y}{\partial x}+xxy=$$

$$C\int\frac{\partial t}{\sqrt{t(1+t)}}\cdot\left(\frac{2Axx}{t}+\frac{2Axx(1+4txx+3ttxx)}{t(1+txx)^{\frac{5}{2}}}\right),$$

quod integrale est

$$\frac{-4ACxx\sqrt{(1+t)}}{\sqrt{t}}+\frac{4ACxx\sqrt{(1+t)}}{(1+txx)^{\frac{3}{2}}\sqrt{t}}$$

$$=\frac{4ACxx\sqrt{(1+t)}}{\sqrt{t}}\left(\frac{1}{(1+txx)^{\frac{3}{2}}}-1\right),$$

et hac forma exprimi potest

$$-2Cx^4\left[3z+x\left(\frac{\partial z}{\partial x}\right)\right]\sqrt{t(1+t)}$$

vel etiam hoc modo

$$-2Cx^4\left(\frac{2A+z}{1+txx}\right)\sqrt{t(1+t)}.$$

Expressio autem ista fit $=0$, primo si $t=-1$, deinde etiam si $t=0$, unde valor pro y inventus

$$y=D\int\frac{\partial t}{t\sqrt{t(1+t)}}\left(1-\frac{1}{\sqrt{(1+txx)}}\right)$$

ita per integrationem definiri debet, ut evanescat posito $t=0$; tum vero ponatur $t=-1$. Vel posito $t=-v$ erit

$$y=D\int\frac{\partial v}{v\sqrt{v(1-v)}}\left(1-\frac{1}{\sqrt{(1-vxx)}}\right)$$

integrali ita sumto ut evanescat posito $v=0$, tum vero facto $v=1$.

Exemplum hoc sufficit ad ostendendum, quomodo constructiones exhibitae aequationi differentio-differentiali satisfaciant; interim vero si quantitas z transcendenter, per logarithmos scilicet exprimitur, consensum nonnisi per calculos nimium taediosos declarare licet.

Problema 136.

1077. Posito $y = C\int (1+t)^{\nu-1} (a+tx)^{\lambda}\, \partial t$, in qua integratione quantitas x ut constans spectatur, integrali per terminos deinceps investigandos definito, ut y aequetur certae functioni ipsius x, invenire aequationes differentio-differentiales formae

$$Lxx \cdot \frac{\partial\partial y}{\partial x^2} + Mx \cdot \frac{\partial y}{\partial x} + Ny = 0,$$

cui ea functio satisfaciat.

Solutio.

Cum sit ex principiis ante stabilitis

$$\frac{\partial y}{\partial x} = C\int \lambda t (1+t)^{\nu-1} (a+tx)^{\lambda-1}\, \partial t \text{ et}$$

$$\frac{\partial\partial y}{\partial x^2} = C\int \lambda(\lambda-1)\, tt (1+t)^{\nu-1} (a+tx)^{\lambda-2}\, \partial t, \text{ erit}$$

$$Lxx \cdot \frac{\partial\partial y}{\partial x^2} + Mx \cdot \frac{\partial y}{\partial x} + Ny = C\int (1+t)^{\nu-1} (a+tx)^{\lambda-2}\, \partial t \times$$

$$[\lambda(\lambda-1)Lttxx + \lambda Mtx(a+tx) + N(a+tx)^2] =$$

$$C\int (1+t)^{\nu-1}(a+tx)^{\lambda-2}\partial t \left\{\begin{matrix} Naa + 2Natx + Nttxx \\ +\lambda Matx + \lambda Mttxx \\ +\lambda(\lambda-1)Lttxx \end{matrix}\right\}$$

quae formula sumta x constante absolute integrabilis esse debet. Ponatur ergo integrale

$$C(1+t)^{\nu}(a+tx)^{\lambda-1}(Paa + Qatx)$$

denotantibus P et Q functionibus quibuscunque ipsius x, erit ejus differentiale

$$C(1+t)^{\nu-1}(a+tx)^{\lambda-2}\partial t \times$$

$$[\nu(a+tx)(Paa+Qatx) + (\lambda-1)x(1+t)(Paa+Qatx) + Qax(1+t)(a+tx)]$$

$$= C(1+t)^{\nu-1}(a+tx)^{\lambda-2}\partial t \left\{\begin{matrix} +\nu Pa^3 & +\nu Qaatx & +\nu Qattxx \\ +(\lambda-1)Paax + \nu Paatx & & +(\lambda-1)Qattxx \\ +Qaax & +(\lambda-1)Paatx & +Qattxx \\ & +(\lambda-1)Qatxx & \\ & +Qaatx & \\ & +Qatxx & \end{matrix}\right\}$$

qua forma cum illa comparata; adipiscimur

$$N = \nu P a + (\lambda - 1) P x + Q x$$
$$2N + \lambda M = (\nu + 1) Q a + (\lambda + \nu - 1) P a + \lambda Q x$$
$$N + \lambda M + \lambda(\lambda - 1) L = (\lambda + \nu) Q a,$$

quarum aequationum extremae demta media praebent

$$\lambda(\lambda-1)L = -(\lambda-1)Pa + (\lambda-1)Px + (\lambda-1)Qa - (\lambda-1)Qx;$$

hincque

$$\lambda L = (a - x)(Q - P) \quad \text{seu} \quad L = \tfrac{1}{\lambda}(a - x)(Q - P),$$

secunda autem demto primae duplo, dat

$$\lambda M = (\lambda - \nu - 1) P a - 2(\lambda - 1) P x + (\nu + 1) Q a + (\lambda - 2) Q x, \quad \text{seu}$$
$$\lambda M = [(\nu + 1) a + (\lambda - 2) x](Q - P) + \lambda(a - x) P.$$

Quare sumtis pro P et Q functionibus quibuscunque ipsius x, si functiones L, M, N ita definiantur, ut sit

$$L = \tfrac{1}{\lambda}(a - x)(Q - P)$$
$$M = \tfrac{1}{\lambda}[(\nu + 1) a + (\lambda + 2) x](Q - P) + (a - x) P$$
$$N = x(Q - P) + (\nu a + \lambda x) P,$$

aequationi differentio-differentiali

$$L x x \partial\partial y + M x \partial x \partial y + N y \partial x^2 = 0$$

satisfaciet formula integralis

$$y = C \int (1 + t)^{\nu - 1} (a + t x)^{\lambda} \partial t,$$

tractata x ut constante, dummodo integrationis termini ita constituantur, ut utroque haec expressio

$$(1 + t)^{\nu} (a + t x)^{\lambda - 1} (P a + Q t x) \text{ evanescat.}$$

Notari autem oportet, hos terminos non ab x pendere debere. Primo autem patet hanc expressionem fieri $= 0$ casu $t = -1$, si modo sit $\nu > 0$. Deinde posito $t = \infty$ etiam evanescet, si modo sit $\nu + \lambda - 1 + 1$ numerus negativus, seu $\nu + \lambda < 0$. Quocirca si sit $\nu > 0$ et $\nu + \lambda < 0$, integrale

$$y = C \int (1 + t)^{\nu - 1} (a + t x)^{\lambda} \partial t$$

ita capi debet, ut posito $t = -1$ evanescat, tum vero statuatur $t = \infty$, functioque ipsius x pro y resultans satisfaciet aequationi propositae.

Corollarium 1.

1078. Quoniam functiones P et Q in formulam integralem pro y assumtam non ingrediuntur, manifestum est eandem formulam satisfacere omnibus aequationibus differentio-differentialibus, quicunque valores litteris P et Q tribuantur.

Corollarium 2.

1079. Sumto ergo $Q = P$ eadem formula integralis

$$y = C \int (1 + t)^{\nu - 1} (a + t)^{\lambda} \partial t$$

satisfacit etiam huic aequationi differentiali primi gradus

$$(a - x) x \partial y + (\nu a + \lambda x) y \partial x = 0.$$

Hujus vero integrale est

$$y = \frac{D (a - x)^{\lambda + n}}{x^{\nu}},$$

qui ergo valor quoque in generae nostrae aequationi differentio-differentiali satisfacit, id quod tentanti mox patebit.

Corollarium 3.

1080. Hic ergo valor integralis

$$y = C \int (1 + t)^{\nu - 1} (a + t x)^{\lambda} \partial t$$

secundum terminos definitos sumtus, congruere debet cum formula algebraica $y = \frac{D (a - x)^{\lambda + \nu}}{x^{\nu}}$, si modo sit $\nu > 0$ et $\lambda + \nu < 0$.

Scholion.

1081. Parum ergo integratio hoc problemate exhibita habet in recessu. Verum reductio formulae integralis

$$y = C\int (1+t)^{\nu-1}(a+tx)^{\lambda}\partial t \text{ ad } y = \frac{D(a-x)^{\lambda+\nu}}{x^{\nu}}$$

eo magis est notatu digna, ad quam illa reducitur, si, integrali ita sumto ut evanescat posito $t = -1$, ponatur $t = \infty$. Posito ergo $\lambda + \nu = -\mu$, ut μ et ν sint numeri positivi, erit

$$C\int \frac{(1+t)^{\nu-1}\partial t}{(a+tx)^{\mu+\nu}} = \frac{D}{x^{\nu}(a-x)^{\mu}}.$$

Vel ponatur $1 + t = z$, erit

$$C\int \frac{z^{\nu-1}\partial z}{(a-x+xz)^{\mu+\nu}} = \frac{D}{x^{\nu}(a-x)^{\mu}},$$

terminis illius integrationis existentibus $z = 0$ et $z = \infty$. Verum etiam haec observatio non magni est momenti, nam posito $a - x = ux$, fit

$$\frac{C}{x^{\mu+\nu}}\int \frac{z^{\nu-1}\partial z}{(u+z)^{\mu+\nu}} = \frac{D}{x^{\mu+\nu}u^{\mu}},$$

ideoque haec formula $\int \frac{z^{\nu-1}\partial z}{(u+z)^{\mu+\nu}}$ ita integrata ut evanescat posito $z = 0$, si tum ponatur $z = \infty$, hanc induet formam $\frac{A}{u^{\mu}}$, in qua A quantitatem constantem denotat ab u non pendentem. Pendet autem ab exponentibus μ et ν, lege ex casibus facile observanda. Scilicet posito

$$\int \frac{z^{\nu-1}\partial z}{(u+z)^{\mu+\nu}} = \frac{A}{u^{\mu}},$$

si sit $\nu = 1$, integrale illud praebet $-\frac{1}{\mu(u+z)^\mu} + \frac{1}{\mu u^\mu}$, et posito $z = \infty$, prodit $\frac{1}{\mu u^\mu}$ ita ut hoc casu sit $A = \frac{1}{\mu}$. Si sit $\nu = 2$, integratio quoque succedit, reperiturque $A = \frac{1}{\mu(\mu+1)}$ si $\nu = 3$ fit $A = \frac{1.2}{\mu(\mu+1)(\mu+2)}$, et si $\nu = 4$ fit

$$A = \frac{1.2.3}{\mu(\mu+1)(\mu+2)(\mu+3)},$$

unde in genere concludimus fore

$$A = \frac{1.2.3. \ldots \ldots (\nu-1)}{\mu(\mu+1)(\mu+2) \ldots \ldots (\mu+\nu-1)}:$$

Quare integratione secundum regulam praescriptam instituta, erit

$$\frac{1}{\mu+1} \cdot \frac{2}{\mu+2} \cdot \frac{3}{\mu+3} \cdots \cdots \frac{\nu-1}{\mu+\nu-1} = \mu u^\mu \int \frac{z^{\nu-1} \partial z}{(u+z)^{\mu+\nu}}.$$

Quod si exponens ν non fuerit integer, valor ipsius A ope interpolationis hujus formulae per factores procedentis definietur. Quadratura scilicet circuli ingredietur si exponens ν fractionem $\frac{1}{2}$ involvat, de hujusmodi autem interpolationibus alibi fusius egimus, neque hic locus est hoc argumentum uberius prosequendi. Restat ultimum hujus sectionis caput, quo aequationum differentio-differentialium integratio per approximationes docebitur.

CAPUT XIII.

DE

AEQUATIONUM DIFFERENTIO-DIFFERENTIALIUM INTEGRATIONE PER APPROXIMATIONES.

Problema 137.

1082.

Proposita aequatione differentio-differentiali quacunque, principia explicare, ex quibus integrationem per approximationes peti oportet.

Solutio

Versetur aequatio proposita inter binas variabiles x et y, ac posito $\partial y = p \partial x$ et $\partial p = q \partial x$, dabitur aequatio inter quatuor quantitates x, y, p et q, ex qua q ita definire licebit, ut q aequetur functioni cuidam trium quantitatum x, y et p, qua vocata $= V$, sit $q = V$, seu $\partial p = V \partial x$. Hic primo observandum est, integrationem, ut sit determinata, duplicem determinationem requirere, seu duas conditiones quasi pro lubitu praescribi posse, quibus satisfaciat. Scilicet non sufficit, ut posito $x = a$ fiat $y = b$, quemadmodum in aequationibus differentialibus primi gradus usu venire vidimus, sed aliam insuper conditionem adjicere licet, quae sit, ut posito $x = a$ fiat etiam $p = \frac{\partial y}{\partial x} = c$ quantitati datae. His ergo determinationibus constitutis, ut posito $x = a$, fiat $y = b$ et $p = c$,

totum integrationis negotium huc reducitur, ut ipsi x alium quemcunque valorem tribuendo, investigentur valores respondentes ipsius y et ipsius p; hoc enim si praestiterimus, aequationis propositae integrale perfecte definiverimus, ut nihil praeterea desiderari possit. Quod cum in genere fieri nequeat, approximationis ratio in hoc consistit, ut ipsi x valor quam minimum ab a discrepans tribuatur, qui sit $x = a + \omega$, et inquiratur, quantum valores quantitatum y et p a primitivis b et c sint discrepaturi. Hic pro principio assumimus, dum x ab a ad $a + \omega$ increscit, etiam quantitatum y et p valores tam parum mutatum iri, ut inde functio V nullam variationem notabilem patiatur. Quare si ponamus, statuendo $x = a$, $y = b$ et $p = c$, fieri $V = F$, eundem valorem F quantitas V retinere censebitur, dum x ab a usque ad $a + \omega$ augetur. Cum igitur pro hoc intervallo minimo habeamus $\partial p = F \partial x$, erit integrando $p = Fx + \text{Const.}$ Verum, quia posito $x = a$, fieri debet $p = c$, erit $p = c + Fx - Fa$. Sit nunc $x = a + \omega$, atque habebimus $p = c + F\omega$, qui est valor ipsius p, valori $x = a + \omega$ respondens. Denique pro hoc minimo intervallo erit $\partial y = c \partial x$, ideoque $y = b + cx - ac$, et pro valore $x = a + \omega$ fit $y = b + c\omega$, qui est valor ipsius y valori $x = a + \omega$ conveniens. Quocirca si valores primitivi sint $x = a$, $y = b$ et $p = \frac{\partial y}{\partial x} = c$, ex iisque fiat $V = F$, sequentes valores intervallo quam minimo ab illis remoti erunt

$$x = a + \omega,\ y = b + c\omega,\ p = c + F\omega,$$

qui si porro ut primitivi spectentur, ex iis simili modo per intervallum quam minimum progredi licet, sicque tandem progressus per intervallum quantumvis magnum innotescet.

Corollarium 1.

1083. Quo minora capiantur haec intervalla, eo minus a vero aberrabitur, dummodo quantitates c et F non sint nimis magnae,

sin autem eae adeo in infinitum excrescant, manifestum est, errorem in quantitatibus y et p insignem commissum iri.

Corollarium 2.

1084. Si quantitas c vel F fiat vehementer magna, intervallum, quo y vel p crescit, pro dato accipi potest; ita posito $c\omega = \psi$ erunt sequentes valores

$$x = a + \frac{\psi}{c},\ y = b + \psi \text{ et } p = c + \frac{F\psi}{c}.$$

At si F prodeat quantitas permagna, valor ipsius p intervallo minimo Φ augeri sumatur, ut sit $F\omega = \Phi$, eruntque valores sequentes $x = a + \frac{\Phi}{F}$, $y = b + \frac{c\Phi}{F}$ et $p = c + \Phi$.

Corollarium 3.

1085. Si b sit quantitas infinita, pro valore proximo ipsius y expediet definiri

$$\frac{1}{y} = \frac{1}{b + c\omega} = \frac{1}{b} - \frac{c\omega}{bb} = \frac{-c\omega}{bb},$$

quae expressio ut sit finita, etiam quantitas c infinita sit oportet; alioquin valor ipsius y respondens non solum ipsi $x = a$ sed etiam ipsi $x = a + \omega$ maneret infinitus.

Scholion 1.

1086. Quoties solutio alicujus problematis pendet ab integratione cujuspiam aequationis differentialis secundi gradus, toties conditiones problematis binas determinationes suppeditare solent; quarum altera, dum ipsi x certus quidam valor $x = a$ tribuitur, exigit ut y datum valorem $y = b$, altera vero ut etiam ratio $\frac{\partial y}{\partial x} = p$ datum valorem $p = c$ consequatur. Quodsi ergo in genere aequationem quandam differentio-differentialem integrare velimus,

integrationem ita instituere licet, ut posito $x = a$, fiat $y = b$ et $p = c$, quantitatibus a, b, c ab arbitrio nostro pendentibus. Interim tamen quandoque usu venire potest, ut posito $x = a$, valores ipsarum y et p non penitus ab arbitrio nostro pendeant, sed ex natura aequationis jam datos valores sortiantur, quibus casibus defectus determinationis aliis conditionibus compensatur. Veluti si proponatur haec aequatio

$$xx(a - bx)\partial\partial y - 2x(2a - bx)\partial x\partial y$$
$$+ 2(3a - bx)y\partial x^2 = 6aa\partial x^2,$$

quomodocunque ea per integrationem determinetur, posito $x = 0$ necessario fit $y = a$ et $\frac{\partial y}{\partial x} = p = b$; ita ut pro casu $x = 0$ valores quantitatum y et p minime arbitrio nostro relinquantur. Integrale autem completum reperitur

$$y = a + bx + \frac{(A + Bx)xx}{a - bx},$$

ubi etiamsi constantes A et B pro lubitu assumantur, tamen semper posito $x = 0$ prodit $y = a$ et $p = b$. Hujusmodi ergo casibus mirum non est, si pro dato ipsius x valore, quantitatum y et p valores arbitrio nostro haud permittantur.

Scholion 2.

1087. Exposita ratio aequationes differentio-differentiales per approximationes integrandi, dum per intervalla minima progredimur, quemadmodum etiam in aequationibus differentialibus primi gradus fecimus, certis casibus difficultatibus involvitur, ut nisi remedium afferatur, in usum vocari nequeat. Primum hoc evenit, quando $c = \infty$, tum enim quantumvis exiguum accipiatur intervallum ω, neque ipsius y neque ipsius p valorem cognoscere licet. Simile quoque incommodum turbat, si positis $x = a$, $y = b$ et $p = c$, functio V fiat infinita, ideoque prodeat $F = \infty$, quo casu valor

ipsius p non definitur. Deinde etiam casus quibus vel c vel F evanescit, seorsim tractari convenit: etsi enim tum valores ipsarum y et p satis accurate ostenduntur, tamen quia nullam mutationem patiuntur, dum mutatio altiore ipsius ω potestate exprimitur, ipsam mutationem investigari utile est, quo in progressu minus a veritate aberretur. Sin autem quantitas b evadat infinita, jam animadvertimus, loco ipsius y ejus reciprocum $\frac{1}{y}$ explorari deberi. Quemadmodum ergo difficultatibus ante memoratis sit occurrendum, diligentius perpendamus.

Problema 138.

1088. Si integrationem per intervalla instituendo, pro initio cujuspiam intervalli, posito $x = a$, $y = b$ et $p = c$ eveniat, ut quantitas c sit vel evanescens vel infinita, integrationem per hoc intervallum absolvere.

Solutio.

Praecedens approximatio dederat $y = b + c(x - a)$, unde si $c = 0$, incrementum ipsius y altiore potestate ipsius $x - a$ exprimetur, scilicet $y = b + A(x - a)^\lambda$, existente $\lambda > 1$, rejectisque altioribus potestatibus, quae prae hac, ob intervallum $x - a$ minimum, recte contemnuntur. Sin autem sit $c = \infty$, valor ipsius y simili modo repraesentari potest, $y = b + A(x - a)^\lambda$, existente $\lambda < 1$, ita tamen ut nihilo sit major; utroque ergo casu eadem investigatio est, ut ex aequatione proposita $\partial p = V \partial x$ tam coëfficiens A quam exponens λ definiatur. Jam ex illa aequatione deducimus

$$\frac{\partial y}{\partial x} = p = \lambda A(x - a)^{\lambda - 1} \text{ et}$$

$$\partial p = \lambda(\lambda - 1) A(x - a)^{\lambda - 2} \partial x,$$

ac necesse est, eandem expressionem resultare, si in formula $V \partial x$ statuatur

$$y = b + A(x-a)^\lambda \text{ et } p = \lambda A (x-a)^{\lambda-1},$$

unde evidens est, fore

$$c = 0 \text{ si } \lambda > 1, \text{ et } c = \infty \text{ si } \lambda < 1.$$

Cum jam V sit functio ipsarum x, y et p, ponatur ubique $x = a$, nisi quatenus formula $x - a$ inest, quae relinquatur, tum vero $y = b$, nisi hinc prodeat $V = 0$ vel $V = \infty$; hoc enim si eveniat, pro y valor $b + A(x-a)^\lambda$ scribatur, similique modo pro p scribatur $\lambda A (x-a)^{\lambda-1}$. Rejiciantur autem formulae $x - a$ potestates altiores prae inferioribus, sicque pro V orietur expressio hujus formae $C(x-a)^\mu$, quae formulae $\lambda(\lambda-1) A (x-a)^{\lambda-2}$ aequari debet, unde tam coëfficiens assumtus A quam exponens λ definietur, ideoque vero proximi valores

$$y = b + A(x-a)^\lambda \text{ et } p = \lambda A (x-a)^{\lambda-1}$$

innotescent, qui eo minus a veritate recedent, quo minor differentia inter a et x constituatur. Casus autem quo $\lambda = 1$ per se est perspicuus, atque in praecedente problemate pertractatus, cum is sit solus quo quantitas c finitum nanciscitur valorem.

Corollarium 1.

1089. Si eveniat ut posito $c = \infty$, quo casu esse debeat $\lambda < 1$, functio V finitum obtineat valorem, cui formula $\lambda(\lambda-1) A (x-a)^{\lambda-2}$ aequari nequit, casus per se nihil habet difficultatis, et valor ipsius y etiam pro minimo excessu ipsius x super a revera infinitus evadet.

Corollarium 2.

1090. Facilius hoc perspicere licet ex exemplo $\partial p = 6 x \partial x$, unde fit

$$p = c + 3 x x - 3 a a = \frac{\partial y}{\partial x}, \text{ hincque}$$

$$y = b + (c - 3 a a)(x - a) + x^3 - a^3, \text{ seu}$$

$$y = b - a c + 2 a^3 + (c - 3 a a) x + x^3.$$

Si ergo constans c sumatur infinita, valor ipsius y semper erit infinitus, solo excepto casu $x = a$.

Corollarium 3.

1091. Sin autem sumto $c = 0$, quo casu esse debet $\lambda > 1$, functio V finitum habeat valorem eumque adeo constantem, posito $x = a$ et $y = b$, ei formula $\lambda (\lambda - 1) A (x - a)^{\lambda - 2}$ aequabitur sumendo $\lambda = 2$, et $2 A =$ illi valori constanti. Veluti in praecedente exemplo fit $V = 6 a = 2 A$, hinc $A = 3 a$, eritque proxime $y = b + 3 a (x - a)^2$, quod etiam congruit cum integrali invento, quod posito $c = 0$ est

$$y = b + 2 a^3 - 3 a a x + x^3 = b + (x + 2 a)(x - a)^2,$$

quae expressio facto $x = a$ in illam abit.

Scholion 1.

1092. Posito $c = 0$ functio V, si in ea scribatur $x = a$, $y = b$ et $p = c = 0$, valorem nanciscetur vel infinite magnum, vel finitum, vel adeo evanescentem. Primo casu, quo fit $V = \infty$, ut ei aequari possit $\lambda (\lambda - 1) A (x - a)^{\lambda - 2}$, sumto $x = a$, necesse est sit $\lambda < 2$, existente $\lambda > 1$; ut autem hinc quantitates A et λ definiantur, in functione V scribi oportet

$$y = b + A (x - a)^\lambda \text{ et } p = \lambda A (x - a)^{\lambda - 1},$$

itemque $x = a$, nisi ubi formula $x - a$ occurrit; hoc modo quia

per hypothesin casu $x = a$ fit $V = \infty$; iste valor prodibit $= C(x-a)^{-\alpha}$, quo collato cum $\lambda(\lambda-1)A(x-a)^{\lambda-2}$ reperientur A et λ, dum ne prodeat $\lambda < 1$, qui casus cum $c = 0$ subsistere nequit. Secundo casu quo prodit V = quantitati finitae, capi oportet $\lambda = 2$, sin autem tertio casu sit $V = 0$, sumi debet $\lambda > 2$, ut ejus valor in formula

$$\lambda(\lambda-1)A(x-a)^{\lambda-2}$$

contineatur. At si debeat esse $c = \infty$, fieri nequit ut vidimus, ut functio V finitum obtineat valorem, multo minus evanescentem, nisi quidem casus incongruos, quibus y perpetuo maneat infinita, admittere velimus. Tum igitur functio V necessario valorem infinitum induit cum formula

$$\lambda(\lambda-1)A(x-a)^{\lambda-2}$$

comparandum, ita ut sit $\lambda < 1$. Hinc igitur patet, determinationem quantitatis c non semper arbitrio nostro relinqui, sed quandoque ex indole ipsius aequationis nobis praescribi. Veluti si proponatur haec aequatio $\partial\partial y = \frac{2\partial x^2}{(x-a)^3}$, erit

$$\frac{\partial y}{\partial x} = p = A - \frac{1}{(x-a)^2} \text{ et } y = B + Ax + \frac{1}{x-a},$$

unde posito

$$x = a \text{ fit } c = A - \frac{1}{0^2} \text{ et } b = B + Aa + \frac{1}{0}, \text{ ergo}$$

$$A = c + \frac{1}{0^2}, \text{ et } B = b - ac - \frac{a}{0^2} + \frac{1}{0},$$

quare, ne aequatio integralis omnino in infinitis versetur, litterae b et c non possunt non esse infinitae.

Scholion 2.

1093. Non autem omnes ordines infinitorum et evanescentium in formula $(x-a)^\lambda$, casu $x = a$, contineri, jam observavimus; expressio scilicet $xxlx$, casu $x = 0$, infinities superat pote-

statem secundam x^2, interim tamen infinities minor est potestate $x^{2-\alpha}$, quantumvis etiam exigua fractio α accipiatur. Quare si in superiori solutione formulas ita instruere velimus, ut ad omnes ordines tam infinitorum quam evanescentium pateant, statui conveniet

$$y = b + A(x-a)^\lambda [l(x-a)]^\mu,$$

unde fit

$$\frac{\partial y}{\partial x} = p = \lambda A (x-a)^{\lambda-1} [l(x-a)]^\mu + \mu A (x-a)^{\lambda-1} [l(x-a)]^{\mu-1};$$

at posito $x = a$, pars prior est ad posteriorem ut $l(x-a)$ ad 1, hoc est ut $\infty : 1$, ex quo sufficit sumsisse

$$p = \lambda A (x-a)^{\lambda-1} [l(x-a)]^\mu,$$

ex quo simili modo colligitur

$$\frac{\partial p}{\partial x} = \lambda(\lambda-1) A (x-a)^{\lambda-2} [l(x-a)]^\mu,$$

quae expressio cum functione V, postquam in ea scripserimus $x = a$, $y = b$ et $p = c$, seu potius

$$y = b + A(x-a)^\lambda [l(x-a)]^\mu \text{ et } p = \lambda A (x-a)^{\lambda-1} [l(x-a)]^\mu,$$

comparari debet, ut inde tam constans A quam exponentes λ et μ innotescant. Haec ergo tenenda sunt in ista investigatione, quo ea ad plures casus extendatur.

Problema 139.

1094. Approximationem ante expositam accuratius persequi, ut sumtis intervallis etiam paulo majoribus minus a vero aberretur.

Solutio.

Posito $\partial y = p \partial x$, aequatio differentio-differentialis hac forma $\frac{\partial p}{\partial x} = V$ exhibeatur, ex qua ante p ita definivimus, quasi V esset quantitas constans pro intervallo saltem vehementer parvo, unde

obtinuimus $p = c + V(x - a)$, postquam scilicet in V posuerimus $x = a$, $y = b$ et $p = c$, qui sunt valores primitivi per intervallum $x - a = \omega$ retinendi. Cum autem functio V interea non sit constans, quia x, y et p involvit, revera erit

$$p = c + V(x - a) - \int (x - a)\, \partial V.$$

Ponamus igitur

$$\partial V = P\, \partial x + Q\, \partial y + R\, \partial p,$$

ut sit

$$\partial V = (P + Q p + R V)\, \partial x,$$

et nunc quantitatem $P + Qp + RV$ ut constantem spectemus, cujus valor prodeat ponendo $x = a$, $y = b$ et $p = c$, quo facto V in F abire supra sumsimus, eritque

$$p = c + F(x - a) - \tfrac{1}{2}(P + Q c + R F)(x - a)^2.$$

Hinc porro ob $\partial y = p\, \partial x$, fit

$$y = b + c(x - a) + \tfrac{1}{2} F(x - a)^2 - \tfrac{1}{6}(P + Q c + R F)(x + a)^3,$$

similique modo approximationem ulterius prosequi licet. Quando autem quantitates P, Q, R et V formulam $x - a$ ejusve potestates complectuntur, quam non amplius ut constantem spectare licet, ejus ratio in integratione est habenda, qua fit ut in seriebus approximantibus formulae $x - a$ poteſtates non ordine ascendant. Tum igitur conveniet pro p ejusmodi seriei initium assumi

$p = c + A(x - a)^{\lambda}$, unde fit

$y = b + c(x - a) + \frac{A}{\lambda + 1}(x - a)^{\lambda + 1}$, et quia

$\frac{\partial p}{\partial x} = \lambda A (x - a)^{\lambda - 1}$,

huic formulae aequari debet functio V, postquam in ea pro y et p valores assumtos, et a pro x scripserimus, nisi formula $x - a$ ingrediatur, hoc modo tam exponens λ quam coëfficiens A determinabitur.

Si c sit $= 0$ vel $= \infty$, ejus ratio potest in calculum introduci, ut ponatur

$$p = f(x - a)^n + A(x - a)^\lambda,$$

unde fit

$$y = b + \frac{f}{n+1}(x - a)^{n+1} + \frac{A}{\lambda+1}(x - a)^{\lambda+1},$$

qui valores si loco x et p substituantur in functione V, prodire debet

$$nf(x - a)^{n-1} + \lambda A(x - a)^{\lambda-1}.$$

Corollarium 1.

1095. Hoc modo per intervalla continuo ulterius progredi licet, dummodo singula non majora accipiantur, quam ut errores commissi maneant insensibiles; atque hac quidem correctione errores illi diminuuntur, ut intervalla etiam majora statui queant.

Corollarium 2.

1096. Pro primo scilicet intervallo valores primitivi $x = a$, $y = b$ et $p = c$ pro lubitu assumuntur, et valores in fine intervalli inventi praebent valores initiales pro secundo intervallo, ex quibus calculus pro hoc intervallo perinde expeditur ac primo; sicque continuo ulterius est progrediendum.

Scholion.

1097. Hujus problematis duplicem solutionem dedimus, quarum prior etsi latissime patere videtur, certis tamen casibus in usum vocari nequit; iis ergo altera solutione uti conveniet. Existunt autem tantum plerumque paucissima ejusmodi intervalla, quae posteriorem methodum postulant, dum reliqua omnia ope prioris

expedire licet. Evenit hoc, quando pro quopiam intervallo quantitates V et c vel evanescunt vel in infinitum excrescunt; quin etiam fieri potest, ut quantumvis exiguum intervallum accipiatur, quantitates y et p variationibus infinitis sint obnoxiae, quarum repraesentatio determinationem prorsus singularem requirit. Veluti si proponatur haec aequatio $\partial\partial y + \frac{y\partial x^2}{xx} = 0$, intervallum ab $x = 0$ usque ad $x = \omega$, etiamsi ω quam minimum assumatur, infinitam mutationem in valoribus y et p indicat; id quod ex ejus integrali completo perspicitur, quod cum sit

$$y = A x^{\frac{1}{2}} \sin. (\tfrac{\sqrt{3}}{2} lx + \alpha),$$

hincque

$$p = \frac{A}{2\sqrt{x}} \sin. (\tfrac{\sqrt{3}}{2} lx + \alpha) + \frac{A\sqrt{3}}{2\sqrt{x}} \cos. (\tfrac{\sqrt{3}}{2} lx + \alpha),$$

seu

$$p = \frac{A}{\sqrt{x}} \sin. (\tfrac{\sqrt{3}}{2} lx + \alpha + 60^{\circ}),$$

evidens est si $x = 0$, fore quidem $y = 0$, sed ipsius p valorem esse incertum. At ipsi x valorem quam minimum tribuendo, y quidem minimum retinebit valorem, sed qui pro minimo intervallo modo sit positivus, modo evanescens, modo negativus, ob maximam mutationem, quam lx patitur, quantitas autem p interea transit per omnes mutationes possibiles. Idem luculentius perspicitur ex hoc exemplo

$$\partial\partial y + \frac{2\partial x \partial y}{x} - \frac{ffy\partial x^2}{x^4} = 0,$$

cujus integrale est $y = A \sin. (\frac{f}{x} + \alpha)$; dum enim x a 0 ad ω crescit, angulus $\frac{f}{x} + \alpha$ ab infinito ad finitum transibit, ejusque sinus interea omnes mutationes ab $+1$ ad -1 infinities adeo subibit. Quando ergo ejusmodi intervalla occurrunt, mirum non est, si consuetae methodi approximandi deficiant, quippe quae hoc principio innituntur, quod mutationes per intervalla minima, sint etiam valde parvae; his autem intervallis exceptis solutio praescripta semper cum usu adhiberi potest.

Exemplum 1.

1098. *Proposita aequatione*

$$\partial \partial y + \frac{y \partial x^2}{fx} = 0,$$

ejus integrationem per approximationem absolvere.

Cum ergo sit $\partial p = -\frac{y \partial x}{fx}$, erit $V = \frac{-y}{fx}$; quare si pro initio intervalli sit $x = a$, $y = b$ et $p = c$, inde tantillum progrediendo, per solutionem priorem, ob

$$P = \frac{y}{fxx} = \frac{-1}{aaf},$$
$$Q = \frac{-1}{fx} = \frac{-1}{af}, \text{ et } R = 0,$$

habebimus

$$p = c - \frac{b}{af}(x - a) - \frac{1}{2}\left(\frac{b}{aaf} - \frac{c}{af}\right)(x - a)^2 \text{ et}$$
$$y = b + c(x - a) - \frac{b}{2aaf}(x - a)^2.$$

Sumto ergo $x - a = \omega$, pro intervallo sequente erunt valores initiales

$$a' = a + \omega,$$
$$b' = b + c\omega - \frac{b\omega^2}{2af}, \text{ et } c' = c - \frac{b\omega}{af} - \frac{(b - ac)\omega\omega}{2aaf},$$

unde simili modo valores initiales pro intervallo sequente colliguntur. Verum si pro quopiam intervallo fiat $a = 0$, operatio peculiari modo institui debet. Posito scilicet pro initio hujus intervalli $x = 0$, $y = b$ et $p = c$, statuatur

$$p = c + Ax^\lambda \text{ et } y = b + cx + \frac{Ax^{\lambda+1}}{\lambda+1}, \text{ erit}$$

$$\frac{\partial p}{\partial x} = \lambda A x^{\lambda-1} = \frac{-y}{fx} = \frac{-b}{fx} - \frac{c}{f} - \frac{Ax^\lambda}{(\lambda+1)f},$$

cui, nisi sit $b = 0$, satisfieri nequit; prodiret enim $\lambda = 0$ et $A = \infty$, unde concludimus poni debere $y = b + Axlx$, ut sit

$p = A\,lx + A$ et $\frac{\partial p}{\partial x} = \frac{A}{x} = \frac{-b - Ax\,lx}{fx}$, hinc $A = -\frac{b}{f}$. Verum quo hinc p accuratius cognoscere liceat, statuamus

$$y = b + A\,x\,lx + Bx, \text{ erit}$$
$$p = A\,lx + A + B \text{ et } \frac{\partial p}{\partial x} = \frac{A}{x},$$

unde concluditur ut ante $A = \frac{-b}{f}$, et B manet indeterminatum, ita ut sit

$$y = b - \frac{bx}{f}\,lx + Bx \text{ et } p = \frac{-b}{f}\,lx - \frac{b}{f} + B,$$

nisi ergo sit $b = 0$, casu $x = 0$, quantitas c necessario est infinita. Quamobrem si intervalli initio sit $x = 0$, $y = b$ et $p = \infty$, pro ejus fine et initio sequentis erit

$$x = \omega,\ y = b - \frac{b\omega}{f}\,l\omega \text{ et } p = \frac{-b}{f}\,l\omega.$$

Exemplum 2.

1099. *Proposita sit haec aequatio*

$$xx\,\partial\partial y - 2x\,\partial x\,\partial y + 2y\,\partial x^2 = \frac{x^2 y\,\partial x^2}{ff},$$

quam per approximationem integrari oporteat.

Cum sit

$$\frac{\partial p}{\partial x} = \frac{2p}{x} - \frac{2y}{xx} + \frac{y}{ff} = V, \text{ erit}$$
$$P = \frac{-2p}{xx} + \frac{4y}{x^3};$$
$$Q = \frac{-2}{xx} + \frac{1}{ff} \text{ et } R = \frac{2}{x}.$$

Hinc si pro cujusque intervalli initio sit $x = a$, $y = b$, $p = c$, ob $F = \frac{2c}{a} - \frac{2b}{aa} + \frac{b}{ff}$, erit

$$p = c + \left(\frac{2c}{a} - \frac{2b}{aa} + \frac{b}{ff}\right)(x - a)$$
$$- \tfrac{1}{2}\left(\frac{c}{ff} + \frac{2b}{aff}\right)(x - a)^2, \text{ et}$$
$$y = b + c(x - a) + \tfrac{1}{2}\left(\frac{2c}{a} - \frac{2b}{aa} + \frac{b}{ff}\right)(x - a)^2;$$

unde calculus per intervalla facile continuatur, dum ne sit $a = 0$. Hoc autem casu quo $a = 0$, difficulter intervallum computo definitur, quia tum fieri nequit, ut quantitatibus b et c dati valores tribuantur, id quod inde facillime intelligitur, quod aequationis propositae integrale completum est

$$y = \text{A}\, e^{\frac{x}{f}} x + \text{B}\, e^{\frac{-x}{f}} x,$$

posito enim $x = 0$, necessario fit $y = 0$, nisi coëfficientes A et B infinitos capere velimus. Sumto autem $b = 0$, approximatio est in promtu.

CALCULI INTEGRALIS
LIBER PRIOR.

PARS SECUNDA,

SEU

METHODUS INVENIENDI FUNCTIONES UNIUS VARIABILIS EX DATA RELATIONE DIFFERENTIALIUM SECUNDI ALTIORISVE GRADUS.

SECTIO POSTERIOR,

DE

RESOLUTIONE AEQUATIONUM DIFFERENTIALIUM TERTII ALTIORUMQUE GRADUUM QUAE DUAS TANTUM VARIABILES INVOLVUNT.

CAPUT I.

DE

INTEGRATIONE FORMULARUM DIFFERENTIALIUM TERTII ALTIORISVE GRADUS SIMPLICIUM.

Problema 140.

1100.

Sumto elemento ∂x constante, invenire integrale completum harum formularum $\partial^3 y = 0$, $\partial^4 y = 0$, $\partial^5 y = 0$ etc. atque in genere hujus $\partial^n y = 0$.

Solutio.

Cum ∂x sit constans, aequatio $\partial^3 y = 0$ per integrationem dat $\partial \partial y = \alpha \partial x^2$, hincque porro integrando $\partial y = \alpha x \partial x + \beta \partial x$, et tandem $y = \frac{1}{2} \alpha x^2 + \beta x + \gamma$.

Simili modo ex aequatione $\partial^4 y = 0$, per quadruplicem integrationem reperitur

1) $\partial^3 y = \alpha \partial x^3$,

2) $\partial \partial y = \alpha x \partial x^2 + \beta \partial x^2$,

3) $\partial y = \frac{1}{2} \alpha x x \partial x + \beta x \partial x + \gamma \partial x$,

et tandem

4) $y = \frac{1}{6} \alpha x^3 + \frac{1}{2} \beta x^2 + \gamma x + \delta$.

Ex aequatione autem $\partial^5 y = 0$, integratio quinquies repetita dat

$$y = \frac{1}{24} \alpha x^4 + \frac{1}{6} \beta x^3 + \frac{1}{2} \gamma x^2 + \delta x + \varepsilon.$$

At hujus aequationis $\partial^6 y = 0$ integrale colligitur

$$y = \tfrac{1}{120}\alpha x^5 + \tfrac{1}{24}\beta x^4 + \tfrac{1}{6}\gamma x^3 + \tfrac{1}{2}\delta x^2 + \varepsilon x + \zeta,$$

sicque ad hujusmodi formas $\partial^n y = 0$, quanticumque fuerint gradus, progredi licet, dummodo n fuerit numerus integer positivus.

Corollarium 1.

1101. A simplicissima forma ergo incipiendo integralia sequenti ordine procedunt

Formularum	Integralia completa sunt
$\partial y = 0$	$y = \alpha$
$\partial \partial y = 0$	$y = \alpha x + \beta$
$\partial^3 y = 0$	$y = \frac{1}{2}\alpha x^2 + \beta x + \gamma$
$\partial^4 y = 0$	$y = \frac{1}{6}\alpha x^3 + \frac{1}{2}\beta x^2 + \gamma x + \delta$
etc.	etc.

Corollarium 2.

1102. Quia constantes α, β, γ etc. ab arbitrio nostro pendent, fractiones tuto rejicere licet, eritque

Formularum	Integralia
$\partial y = 0$	$y = \alpha$
$\partial \partial y = 0$	$y = \alpha x + \beta$
$\partial^3 y = 0$	$y = \alpha x^2 + \beta x + \gamma$
$\partial^4 y = 0$	$y = \alpha x^3 + \beta x^2 + \gamma x + \delta$
$\partial^5 y = 0$	$y = \alpha x^4 + \beta x^3 + \gamma x^2 + \delta x + \varepsilon$
etc.	etc.

Corollarium 3.

1103. Quoti ergo ordinis est formula differentialis, totidem constantes arbitrarias ejus integrale completum complectitur, quas pro quovis casu oblato secundum conditiones praescriptas definiri oportet.

Scholion 1.

1104. Ponendo $\partial y = p\,\partial x$; $\partial p = q\,\partial x$, $\partial q = r\,\partial x$, $\partial r = s\,\partial x$ etc. omnes aequationes differentiales altiorum ordinum ad quantitates finitas reducuntur, in quibus nulla amplius ratio ejus elementi, quod constans assumitur, habetur. Atque hinc formae omnium aequationum differentialium sequenti modo repraesentari possunt:

Aequationum differentialium	Forma generalis
I. gradus	$p = f:(x \text{ et } y)$
II. gradus	$q = f:(x,\ y,\ \text{et } p)$
III. gradus	$r = f:(x,\ y,\ p,\ \text{et } q)$
IV. gradus	$s = f:(x,\ y,\ p,\ q,\ \text{et } r)$
	etc.

ubi quantitates y, p, q, r, s etc. ita, excludendo ∂x, a se invicem pendent, ut cum sit

$$\partial x = \frac{\partial y}{p} = \frac{\partial p}{q} = \frac{\partial q}{r} = \frac{\partial r}{s} = \frac{\partial s}{t} \text{ etc.}$$

sequentes relationes locum habeant

$$q\,\partial y = p\,\partial p,\quad r\,\partial y = p\,\partial q,\quad s\,\partial y = p\,\partial r,\quad t\,\partial y = p\,\partial s \text{ etc.}$$
$$r\,\partial p = q\,\partial q,\quad s\,\partial p = q\,\partial r,\quad t\,\partial p = q\,\partial s \text{ etc.}$$
$$s\,\partial q = r\,\partial r,\quad t\,\partial q = r\,\partial s \text{ etc.}$$
$$t\,\partial r = s\,\partial s \text{ etc.}$$

quarum formularum quaedam per se sunt integrabiles veluti

$$\int q\,\partial y = \tfrac{1}{2}pp,\ \int r\,\partial p = \tfrac{1}{2}qq,\ \int s\,\partial q = \tfrac{1}{2}rr,\ \int t\,\partial r = \tfrac{1}{2}ss \text{ etc.}$$

ex quibus porro ob $\int z\,\partial v = vz - \int v\,\partial z$ istae concluduntur

$$\int y\,\partial q = yq - \tfrac{1}{2}pp,\ \int p\,\partial r = pr - \tfrac{1}{2}qq,\ \int q\,\partial s = qs - \tfrac{1}{2}rr,\ \int r\,\partial t = rt - \tfrac{1}{2}ss, \text{ etc.}$$

quarum ope ex praecedentibus deducitur

$$\int s\,\partial y = pr - \tfrac{1}{2}qq, \text{ hinc } \int y\,\partial s = ys - pr + \tfrac{1}{2}qq,$$

$$\int t\,\partial p = qs - \tfrac{1}{2}rr, \text{ hinc } \int p\,\partial t = pt - qs + \tfrac{1}{2}rr,$$
$$\int u\,\partial q = rt - \tfrac{1}{2}ss, \text{ hinc } \int q\,\partial u = qu - rt + \tfrac{1}{2}ss.$$

Hinc porro definitur $\int y\,\partial u = yu - \int u\,\partial y$, at $\frac{\partial y}{p} = \frac{\partial t}{u}$, unde

$$\int y\,\partial u = yu - \int p\,\partial t = yu - pt + qs - \tfrac{1}{2}rr.$$

Quare si differentialia iterum introducamus, obtinebimus sequentes formulas integrales

$$\int y\,\partial y = \tfrac{1}{2}xy$$
$$\int y\,\partial^3 y = y\,\partial\partial y - \tfrac{1}{2}\partial y^2$$
$$\int y\,\partial^5 y = y\,\partial^4 y - \partial y\,\partial^3 y + \tfrac{1}{2}\partial\partial y^2$$
$$\int y\,\partial^7 y = y\,\partial^6 y - \partial y\,\partial^5 y + \partial\partial y\,\partial^4 y - \tfrac{1}{2}\partial^3 y^2$$
etc.

ita ut formula $\int y\,\partial^n y$ sit integrabilis, quoties n est numerus impar.

Scholion 2.

1105. In aequationibus differentialibus secundi gradus formas simpliciores ita constituimus, ut q aequetur functioni vel ipsius x tantum, vel ipsius y, vel ipsius p, quas litteras majusculas pro functionibus minuscularum scribendo ita repraesentare licet, ut sit

vel $q = X$, vel $q = Y$, vel $q = P$.

Hinc simili modo pro aequationibus differentialibus tertii gradus formas simpliciores constituere possumus

$r = X$, $r = Y$, $r = P$, $r = Q$,

ita ut tantum binas quantitates variabiles involvant.

Pro quarto autem gradu essent formae simpliciores

$s = X$, $s = Y$, $s = P$, $s = Q$, $s = R$,

et pro quinto

$t = X$, $t = Y$, $t = P$, $t = Q$, $t = R$, $t = S$,

atque ita porro pro superioribus.

Verum hae formae non omnes aeque integrationem admittunt, dum aliae ne semel quidem, aliae semel tantum, aliae per omnes integrationes usque ad relationem inter x et y perduci possunt, cujusmodi sunt primae quaeque in quovis gradu. Semper autem id est propositum, ut relatio inter binas variabiles principales x et y eliciatur.

Problema 141.

1106. Posito $\partial y = p\,\partial x$, $\partial p = q\,\partial x$, $\partial q = r\,\partial x$, $\partial r = s\,\partial x$, $\partial s = t\,\partial x$, etc. pro quovis differentialium gradu, si litterarum p, q, r, s, t, etc. quaepiam aequetur functioni ipsius x, quae sit X, invenire relationem inter x et y.

Solutio.

Si primo sit $p = X$, per ∂x multiplicando erit $p\,\partial x = \partial y = X\,\partial x$ hincque $y = \int X\,\partial x$, qui est casus formularum differentialium primi gradus simplicium.

Sit secundo $q = X$, erit $q\,\partial x = \partial p = X\,\partial x$, hinc $p = \int X\,\partial x$, et $p\,\partial x = \partial y = \partial x \int X\,\partial x$, ergo $y = \int \partial x \int X\,\partial x$, seu per simplicia integralia $y = x \int X\,\partial x - \int X x\,\partial x$.

Sit tertio $r = X$, ob $\partial q = r\,\partial x$ erit $q = \int X\,\partial x$, hincque

$$p = \int q\,\partial x = \int \partial x \int X\,\partial x = x \int X\,\partial x - \int X x\,\partial x,$$

ac tandem

$$y = \int p\,\partial x = \int \partial x \int \partial x \int X\,\partial x = \tfrac{1}{2} x x \int X\,\partial x - x \int X x\,\partial x + \tfrac{1}{2} \int X x x\,\partial x.$$

Sit quarto $s = X$, ac reperitur $y = \int \partial x \int \partial x \int \partial x \int X\,\partial x$, quae expressio evolvitur in hanc

$$y = \tfrac{1}{6} x^3 \int X\,\partial x - \tfrac{1}{2} x x \int X x\,\partial x + \tfrac{1}{2} x \int X x x\,\partial x - \tfrac{1}{6} \int X x^3\,\partial x.$$

Sit quinto $t = X$, erit $y = \int \partial x \int \partial x \int \partial x \int \partial x \int X \partial x$, seu

$$y = \tfrac{1}{24} x^4 \int X \partial x - \tfrac{1}{6} x^3 \int X x \partial x + \tfrac{1}{4} x^2 \int X x x \partial x - \tfrac{1}{6} x \int X x^3 \partial x + \tfrac{1}{24} \int X x^4 \partial x;$$

unde lex ulterius progrediendi est manifesta.

Corollarium 1.

1107. Tot ergo habentur formulae integrales, quoti gradus aequatio fuerit differentialis, et quia quaelibet constantem arbitrariam assumit, totidem constantes in integrale ingrediuntur, quibus id completum redditur; quod idem ex priori forma, ubi totidem signa integralia implicantur, intelligitur.

Corollarium 2.

1108. Sumto elemento ∂x constante, sequentium formularum more consueto expressarum integralia completa ita se habebunt

I. Si $\partial y = X \partial x$ est $y = \int X \partial x$,

II. Si $\partial \partial y = X \partial x^2$ est $1 y = x \int X \partial x - \int X x \partial x$,

III. Si $\partial^3 y = X \partial x^3$ est
$$2 y = x^2 \int X \partial x - 2 x \int X x \partial x + \int X x^2 \partial x,$$

IV. Si $\partial^4 y = X \partial x^4$ est
$$6 y = x^3 \int X \partial x - 3 x^2 \int X x \partial x + 3 x \int X x^2 \partial x - \int X x^3 \partial x,$$

V. Si $\partial^5 y = X \partial x^5$ est
$$24 y = x^4 \int X \partial x - 4 x^3 \int X x \partial x + 6 x^2 \int X x^2 \partial x - 4 x \int X x^3 \partial x + \int X x^4 \partial x,$$

etc.

Scholion.

1109. Formulas autem, quas supra secundo loco constituimus, functionem Y complectentes, post secundum gradum integrare

non licet. Ex tertio enim ordine formula $r = Y$ etsi novimus esse

$$r = \frac{p\,\partial q}{\partial y} = \frac{q\,\partial q}{\partial p} = \frac{\partial q}{\partial x},$$

nullo modo integrari potest; neque etiam hinc q per y determinari potest. Nam sumta forma $p\,\partial q = Y\,\partial y$, existente $p\,\partial p = q\,\partial y$ ob $p = \frac{Y\,\partial y}{\partial q}$, erit

$$\partial p = \frac{\partial y\,\partial Y}{\partial q} + Y\,\partial\,.\,\frac{\partial y}{\partial q},$$

hincque p elidendo

$$\frac{Y\,\partial y^2\,\partial Y}{\partial q^2} + \frac{YY\,\partial y}{\partial q}\,\partial\,.\,\frac{\partial y}{\partial q} = q\,\partial y,$$

quae quidem aequatio est secundi gradus, sed neutiquam in genere resolutionem admittit. Ex quarto genere formula $s = Y$, ob $\int s\,\partial y = p\,r - \frac{1}{2}qq = \int Y\,\partial y$, semel integrari potest, sed hinc ulterius progredi non licet. Quas autem supra pro quovis gradu formulas simpliciores ultimo loco constituimus itemque penultimo, eae tractabiles deprehenduntur: earum ergo integrationem investigemus.

Problema 142.

1110. Posito ut hactenus $\partial y = p\,\partial x$, $\partial p = q\,\partial x$, $\partial q = r\,\partial x$, etc. litterae Y, P, Q, R, etc. denotent functiones cujusque litterae minusculae cognominis, investigare integralia harum formularum simplicium

$$p = Y,\quad q = P,\quad r = Q,\quad s = R,\quad t = S,\ \text{etc.}$$

Solutio.

Aequatio prima ob $p = \frac{\partial y}{\partial x}$ statim dat $\partial x = \frac{\partial y}{Y}$, ideoque $x = \int\frac{\partial y}{Y}$.

Aequatio secunda $q = P$ ob $q = \frac{\partial p}{\partial x}$ praebet $\partial x = \frac{\partial p}{P}$, et $\partial y = \frac{p\,\partial p}{P}$, unde cum P sit functio ipsius p, utraque variabilis x

et y per p determinatur hoc modo

$$x = \int \frac{\partial p}{P} \text{ et } y = \int \frac{p \partial p}{P}.$$

Aequatio tertia $r = Q$, ob $r = \frac{\partial q}{\partial x}$, dat $\partial x = \frac{\partial q}{Q}$, hinc $q \partial x = \partial p = \frac{q \partial q}{Q}$, ita ut sit $x = \int \frac{\partial q}{Q}$ et $p = \int \frac{q \partial q}{Q}$ unde colligimus $p \partial x = \partial y = \frac{\partial q}{Q} \int \frac{q \partial q}{Q}$, ergo $y = \int \frac{\partial q}{Q} \int \frac{q \partial q}{Q}$. Quare per eandem variabilem q utraque variabilis x et y ita determinatur, ut sit

$$x = \int \frac{\partial q}{Q} \text{ et } y = \int \frac{\partial q}{Q} \int \frac{q \partial q}{Q}.$$

Aequatio quarta $s = R = \frac{\partial r}{\partial x}$ dat $\partial x = \frac{\partial r}{R}$, unde colligimus $r \partial x = \partial q = \frac{r \partial r}{R}$, ita ut sit $q = \int \frac{r \partial r}{R}$. Porro $q \partial x = \partial p$ dat $\partial p = \frac{\partial r}{R} \int \frac{r \partial r}{R}$, hincque $p = \int \frac{\partial r}{R} \int \frac{r \partial r}{R}$, et quia $p \partial x = \partial y$ habebimus $\partial y = \frac{\partial r}{R} \int \frac{\partial r}{R} \int \frac{r \partial r}{R}$, quare per r ambae variabiles principales x et y ita definiuntur

$$x = \int \frac{\partial r}{R} \text{ et } y = \int \frac{\partial r}{R} \int \frac{\partial r}{R} \int \frac{r \partial r}{R}.$$

Aequatio quinta $t = S$ simili modo tractata praebet

$$x = \int \frac{\partial s}{S} \text{ et } y = \int \frac{\partial s}{S} \int \frac{\partial s}{S} \int \frac{\partial s}{S} \int \frac{s \partial s}{S}$$

sicque facile ulterius progredi licet.

Corollarium 1.

1111. Ex formula secunda intelligitur, si x aequetur functioni ipsius p, ut sit $x = P$, fore $y = \int p \partial P = Pp - \int P \partial p$, quod quidem per se est manifestum.

Corollarium 2.

1112. Sin autem sit $x = Q$, ob $\partial x = \partial Q$ erit

$$q \partial x = \partial p = q \partial Q, \text{ et } p = \int q \partial Q,$$

hincque

$$y = \int \partial Q \int q \partial Q, \text{ seu } y = Q \int q \partial Q - \int q Q \partial Q.$$

Vel etiam cum sit

$$y = \int \partial Q (qQ - \int Q \partial q), \text{ erit } y = \tfrac{1}{2} qQQ + \tfrac{1}{2} \int QQ\partial q - Q\int Q\partial q,$$

sive hoc modo

$$2y = QQq - 2Q\int Q\partial q - \int QQ\partial q.$$

Corollarium 3.

1113. Simili modo si $x = R$, erit

$$q = \int r\partial x = \int r\partial R \text{ et } p = \int q\partial x = \int \partial R \int r\partial R,$$

atque

$$y = \int p\partial x = \int \partial R \int \partial R \int r\partial R, \text{ seu}$$
$$2y = \int \partial R (RRr - 2R\int R\partial r + \int RR\partial r),$$

per praecedens Corollarium. Ergo per similes reductiones

$$6y = R^3 r - 3R^2 \int R\partial r + 3R\int RR\partial r - \int R^3 \partial r.$$

Corollarium 4.

1114. At si fuerit $x = S$, reperietur per similes reductiones

$$24y = S^4 s - 4S^3 \int S\partial s + 6S^2 \int SS\partial s - 4S\int S^3 \partial s + \int S^4 \partial s:$$

hinc ergo per differentiationes retrogrediendo

$$24p\partial S = 4S^3 s\partial S - 12SS\partial S\int S\partial s + 12S\partial S\int SS\partial s - 4\partial S\int S^3 \partial s,$$

seu

$$6p = S^3 s - 3SS\int S\partial s + 3S\int SS\partial s - \int S^3 \partial s, \text{ et}$$
$$2q = S^2 s - 2S\int S\partial s + \int SS\partial s,$$

tum $r = Ss - \int S\partial s$ et $s = s$.

Problema 143.

1115. Iisdem manentibus denominationibus, quibus hactenus sumus usi, investigare integralia harum formularum simpliciorum

$$q = Y, \; r = P, \; s = Q, \; t = R, \text{ etc.}$$

Solutio.

Pro formula prima $q=Y$, cum sit $q=\frac{p\partial p}{\partial y}$, erit $p\partial p=Y\partial y$ et $pp=2\int Y\partial y$, hinc $p=\sqrt{2\int Y\partial y}=\frac{\partial y}{\partial x}$, unde colligitur $x=\int\frac{\partial y}{\sqrt{2\int Y\partial y}}$, sicque x per y determinatur.

Pro formula secunda $r=P$ ob $r=\frac{q\partial q}{\partial p}$, habebimus $q\partial q=P\partial p$ et $q=\sqrt{2\int P\partial p}=\frac{p\partial p}{\partial y}=\frac{\partial p}{\partial x}$, unde concludimus

$$x=\int\frac{\partial p}{\sqrt{2\int P\partial p}}, \text{ et } y=\int\frac{p\partial p}{\sqrt{2\int P\partial p}}.$$

Pro formula tertia $s=Q$ ob $s=\frac{r\partial r}{\partial q}$, fiet

$$r=\sqrt{2\int Q\partial q}=\frac{q\partial q}{\partial p},$$

unde sequitur $p=\int\frac{q\partial q}{\sqrt{2\int Q\partial q}}$. Cum vero sit $r=\frac{\partial q}{\partial x}$, erit $\partial x=\frac{\partial q}{\sqrt{2\int Q\partial q}}$, et ob $p\partial x=\partial y$ habebimus

$$x=\int\frac{\partial q}{\sqrt{2\int Q\partial q}}, \text{ et } y=\int\frac{\partial q}{\sqrt{2\int Q\partial q}}\int\frac{q\partial q}{\sqrt{2\int Q\partial q}}.$$

Pro formula quarta $t=R$ ob $t=\frac{s\partial s}{\partial r}$, nanciscimur $s=\sqrt{2\int R\partial r}$. At est $s=\frac{\partial r}{\partial x}$, unde fit $\partial x=\frac{\partial r}{\sqrt{2\int R\partial r}}$. Est vero etiam $s=\frac{r\partial r}{\partial q}$, ideoque $q=\int\frac{r\partial r}{\sqrt{2\int R\partial r}}$; sed quoniam $p=\int q\partial x$, fit $p=\int\frac{\partial r}{\sqrt{2\int R\partial r}}\int\frac{r\partial r}{\sqrt{2\int R\partial r}}$, ex quo prodit $y=\int p\partial x$. Quocirca x et y ita per r determinantur, ut sit

$$x=\int\frac{\partial r}{\sqrt{2\int R\partial r}}, \text{ et } y=\int\frac{\partial r}{\sqrt{2\int R\partial r}}\int\frac{\partial r}{\sqrt{2\int R\partial r}}\int\frac{r\partial r}{\sqrt{2\int R\partial r}}.$$

Pro formula quinta $u=S$, ob $u=\frac{t\partial t}{\partial s}$, adipiscimur $t=\sqrt{2\int S\partial s}=\frac{\partial s}{\partial x}$, ut sit $\partial x=\frac{\partial s}{\sqrt{2\int S\partial s}}$. Est vero etiam $t=\frac{s\partial s}{\partial r}$, ergo $r=\int\frac{s\partial s}{\sqrt{2\int S\partial s}}$. Tum $q=\int r\partial x$, $p=\int q\partial x$ et $y=\int p\partial x$, ex quibus conficitur

$$x=\int\frac{\partial s}{\sqrt{2\int S\partial s}} \text{ et } y=\int\frac{\partial s}{\sqrt{2\int S\partial s}}\int\frac{\partial s}{\sqrt{2\int S\partial s}}\int\frac{\partial s}{\sqrt{2\int S\partial s}}\int\frac{s\partial s}{\sqrt{2\int S\partial s}}.$$

Scholion.

1116. Hi sunt casus, quibus formulas illas simpliciores supra recensitas resolvere licet, hincque methodus patet, qua reliquae tractari queant. Multo pauciores occurrunt casus tractabiles in formis magis compositis, ubi $\frac{\partial^n y}{\partial x^n}$ aequatur functioni binarum pluriumve quantitatum variabilium; ob quam penuriam parum admodum suppetit, quod in hac sectione exponere queamus. Aequationum autem, quae per methodos adhuc inventas tractari possunt, haec fere est forma generalis

$$Ay + B . \frac{\partial y}{\partial x} + C . \frac{\partial \partial y}{\partial x^2} + D . \frac{\partial^3 y}{\partial x^3} + \text{etc.} = 0,$$

sumto elemento ∂x constante, quae etiam ponendo

$$\partial y = p \partial x, \ \partial p = q \partial x, \ \partial q = r \partial x, \ \text{etc.}$$

ita repraesentari potest

$$Ay + Bp + Cq + Dr + Es + \text{etc.} = 0.$$

Deinde vero etiam aequationes hac forma latius patente contentae resolutionem admittunt

$$Ay + Bp + Cq + Dr + Es + \text{etc.} = X,$$

denotante X functionem quamcunque ipsius x. Porro quoque sequentes formae, quae quidem ad illas reduci possunt

$$Ay + \frac{Bp}{x} + \frac{Cq}{xx} + \frac{Dr}{x^3} + \frac{Es}{x^4} + \text{etc.} = 0 \text{ et}$$

$$Ay + \frac{Bp}{x} + \frac{Cq}{xx} + \frac{Dr}{x^3} + \frac{Es}{x^4} + \text{etc.} = X,$$

quarum resolutio adeo succedit, ad quantumvis gradum etiam differentialitas assurgat; in harum ergo aequationum evolutione tractatio nostra versabitur.

CAPUT II.

DE RESOLUTIONE AEQUATIONUM HUJUS FORMAE

$$Ay + B.\frac{\partial y}{\partial x} + C.\frac{\partial\partial y}{\partial x^2} + D.\frac{\partial^3 y}{\partial x^3} + E.\frac{\partial^4 y}{\partial x^4} + \text{etc.} = 0,$$

SUMTO ELEMENTO ∂x CONSTANTE.

Problema 144.

1117.

Aequationis differentialis tertii gradus

$$Ay + \frac{B\partial y}{\partial x} + \frac{C\partial\partial y}{\partial x^2} + \frac{D\partial^3 y}{\partial x^3} = 0,$$

sumto elemento ∂x constante, integrale completum invenire.

Solutio

Cum A, B, C, D sint quantitates constantes, levi attentione adhibita patet, isti aequationi hujusmodi formam $y = e^{\lambda x}$ satisfacere; cum enim hinc sit

$$\frac{\partial y}{\partial x} = \lambda e^{\lambda x}, \quad \frac{\partial\partial y}{\partial x^2} = \lambda^2 e^{\lambda x}, \quad \frac{\partial^3 y}{\partial x^3} = \lambda^3 e^{\lambda x},$$

his substitutis et aequatione per $e^{\lambda x}$ divisa, fit

$$A + \lambda B + \lambda^2 C + \lambda^3 D = 0,$$

unde exponens λ determinatur, qui cum tres valores sortiatur, qui sint α, β, γ, habebimus tres formulas satisfacientes $y = e^{\alpha x}$, $y = e^{\beta x}$, $y = e^{\gamma x}$. Verum ex natura aequationis propositae perspicuum est, si ei satisfaciant valores $y = P$, $y = Q$, $y = R$, etiam his utcunque conjungendis satisfacturam

$$y = \mathfrak{A}P + \mathfrak{B}Q + \mathfrak{C}R.$$

Quare ex ternis formulis inventis nanciscimur hanc latissime patentem aeque satisfacientem

$$y = \mathfrak{A} e^{\alpha x} + \mathfrak{B} e^{\beta x} + \mathfrak{C} e^{\gamma x},$$

quae forma cum tres constantes arbitrarias $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ complectatur, revera erit integrale completum aequationis nostrae propositae.

Corollarium 1.

1118. Integrale ergo completum tot constat partibus, quot radices habeat seu factores aequatio

$$A + B\lambda + C\lambda^2 + D\lambda^3 = 0,$$

cujus si factor fuerit $a + \lambda$, pars integralis erit e^{-ax}.

Corollarium 2.

1119. Haec scilicet pars erit integrale hujus aequationis $ay + \frac{\partial x}{\partial y} = 0$. Unde si sit

$$A + B\lambda + C\lambda^2 + D\lambda^3 = (a+\lambda)(b+\lambda)(c+\lambda),$$

quaerantur valores ipsius y ex his aequationibus simplicibus

$$ay + \frac{\partial y}{\partial x} = 0, \quad by + \frac{\partial y}{\partial x} = 0, \quad cy + \frac{\partial y}{\partial x} = 0,$$

qui si sint $y = P$, $y = Q$, $y = R$, integrale aequationis propositae erit

$$y = \mathfrak{A}P + \mathfrak{B}Q + \mathfrak{C}R.$$

Corollarium 3.

1120. Si binae radices sint aequales, puta $\beta = \alpha$, consideretur differentia ut evanescens vel $\beta = \alpha + \omega$, et cum sit

$$e^{\beta x} = e^{\alpha x} \cdot e^{\omega x} = e^{\alpha x}(1 + \omega x),$$

evidens est loco $\mathfrak{A} e^{\alpha x} + \mathfrak{B} e^{\beta x}$ scribi debere

$$\mathfrak{A} e^{\alpha x} + \mathfrak{B} e^{\alpha x} x = e^{\alpha x}(\mathfrak{A} + \mathfrak{B} x).$$

Ac, si omnes tres radices fuerint aequales $\alpha = \beta = \gamma$, ut aequatio sit

$$y + \frac{3\partial y}{a\partial x} + \frac{3\partial\partial y}{a^2\partial x^2} + \frac{\partial^3 y}{a^3\partial x^3} = 0$$

ob $\alpha = \beta = \gamma = -a$, integrale completum erit

$$e^{-ax}(\mathfrak{A} + \mathfrak{B}x + \mathfrak{C}xx).$$

Corollarium 4.

1121. Si binae radices fiant imaginariae, ut fit

$$\alpha = \mu + \nu\sqrt{-1} \text{ et } \beta = \mu - \nu\sqrt{-1},$$

loco $\mathfrak{A}e^{\alpha x} + \mathfrak{B}e^{\beta x}$ scribi debet

$$e^{\mu x}(\mathfrak{A}e^{\nu x\sqrt{-1}} + \mathfrak{B}e^{-\nu x\sqrt{-1}}),$$

quae reducitur ad hanc formam

$$e^{\mu x}(\mathfrak{A}\cos.\nu x + \mathfrak{B}\sin.\nu x).$$

Scholion 1.

1122. Quamquam aequatio proposita triplicem integrationem postulat, antequam ad relationem finitam inter x et y perveniatur, hic tamen una operatione, quae ne integrationi quidem est affinis, eo pertigimus. Conjectura scilicet formam collegimus aequationi particulariter satisfacientem, simulque tres hujusmodi formas sumus consecuti. Deinde ex ipsa aequationis indole intelleximus, si valores singuli $y = P$, $y = Q$, $y = R$ satisfaciant, etiam formam ex iis compositam $y = \mathfrak{A}P + \mathfrak{B}Q + \mathfrak{C}R$ satisfacere debere, quod nisi commode evenisset, ex illis ternis valoribus nihil amplius concludi potuisset. Ex eodem ergo principio in genere hujusmodi aequationes differentiales, quotcunque fuerint gradus, uno quasi actu ita resolvi poterunt, ut adeo integrale completum assignetur.

Scholion 2.

1123. Quoniam aequationem differentialem tertii gradus

$$Ay + B\frac{\partial y}{\partial x} + C\frac{\partial\partial y}{\partial x^2} + D\frac{\partial^3 y}{\partial x^3} = 0,$$

in genere resolvere licuit, ut integrale completum esset

$$y = \mathfrak{A}\, e^{\alpha x} + \mathfrak{B}\, e^{\beta x} + \mathfrak{C}\, e^{\gamma x},$$

existentibus α, β, γ radicibus hujus aequationis cubicae

$$A + B\lambda + C\lambda^2 + D\lambda^3 = 0,$$

hinc usum non contemnendum pro aliis aequationibus, in quas illam transformare licet, percipiemus. Primo autem illam aequationem ad differentialem secundi gradus revocare licet ope substitutionis $y = e^{\int u\partial x}$; unde fit

$$\frac{\partial y}{\partial x} = e^{\int u\partial x}u, \quad \frac{\partial\partial y}{\partial x^2} = e^{\int u\partial x}\left(\frac{\partial u}{\partial x} + uu\right), \text{ et}$$

$$\frac{\partial^3 y}{\partial x^3} = e^{\int u\partial x}\left(\frac{\partial\partial u}{\partial x^2} + \frac{3u\partial u}{\partial x} + u^3\right),$$

ita ut aequatio transformata sit, divisione per $e^{\int u\partial x}$ facta,

$$A + Bu + Cuu + Du^3 + C\frac{\partial u}{\partial x} + \frac{3Du\partial u}{\partial x} + \frac{D\partial\partial u}{\partial x^2} = 0,$$

cujus ergo ob $u = \frac{\partial y}{y\partial x}$ integrale est

$$u = \frac{\alpha\mathfrak{A}\, e^{\alpha x} + \beta\mathfrak{B}\, e^{\beta x} + \gamma\mathfrak{C}\, e^{\gamma x}}{\mathfrak{A}\, e^{\alpha x} + \mathfrak{B}\, e^{\beta x} + \mathfrak{C}\, e^{\gamma x}}.$$

Haec autem aequatio ulterius ad gradum primum reducitur ponendo $\partial x = \frac{\partial u}{t}$, cum enim elementum ∂x sumtum sit constans, erit $t\partial\partial u - \partial t\partial u = 0$, seu $\partial\partial u = \frac{\partial t\partial u}{t}$, unde fit $\frac{\partial u}{\partial x} = t$, et $\frac{\partial\partial u}{\partial x^2} = \frac{t\partial t}{\partial u}$, ita ut prodeat haec aequatio differentialis primi gradus

$$A + Bu + Cuu + Du^3 + t(C + 3Du) + \frac{Dt\partial t}{\partial u} = 0,$$

cujus ergo resolutio quoque est in potestate: utriusque scilicet variabilis u et t valor per eandem variabilem x exprimi potest.

Cum enim y per x detur, erit primo $u = \frac{\partial y}{y \partial x}$; tum vero $t + uu = \frac{\partial\partial y}{y \partial x^2}$, ob $\frac{\partial u}{\partial x} = t$. Loco y ergo substituto valore supra invento erit

$$u = \frac{\alpha \mathfrak{A} e^{\alpha x} + \beta \mathfrak{B} e^{\beta x} + \gamma \mathfrak{C} e^{\gamma x}}{\mathfrak{A} e^{\alpha x} + \mathfrak{B} e^{\beta x} + \mathfrak{C} e^{\gamma x}}, \text{ et}$$

$$t + uu = \frac{\alpha \alpha \mathfrak{A} e^{\alpha x} + \beta \beta \mathfrak{B} e^{\beta x} + \gamma \gamma \mathfrak{C} e^{\gamma x}}{\mathfrak{A} e^{\alpha x} + \mathfrak{B} e^{\beta x} + \mathfrak{C} e^{\gamma x}},$$

dummodo α, β, γ sint radices ex hac aequatione

$$A + B\lambda + C\lambda^2 + D\lambda^3 = 0.$$

Observari autem convenit, illam aequationem, posito $t + uu = z$, abire in hanc formam

$$A + Bu + z(C + Du) + \frac{D \partial z}{\partial u}(z - uu) = 0,$$

quae latius patere videtur, quam illae ejusdem generis aequationes, quas supra tractavimus; cujus, quia ratio per methodos cognitas integrandi non constat, resolutio facillime instituitur ponendo

$$u = \frac{\partial y}{y \partial x} \text{ et } z = \frac{\partial\partial y}{y \partial x^2},$$

unde fit

$$\partial u = \frac{\partial\partial y}{y \partial x} - \frac{\partial y^2}{yy \partial x} \text{ et } \partial z = \frac{\partial^3 y}{y \partial x^2} - \frac{\partial y \partial\partial y}{yy \partial x^2},$$

ideoque

$$\frac{\partial z}{\partial u} = \frac{y \partial^3 z - \partial y \partial\partial y}{\partial x (y \partial\partial y - \partial y^2)} \text{ et } z - uu = \frac{y \partial\partial y - \partial y^2}{yy \partial x^2},$$

sicque resultat haec aequatio

$$A + \frac{B \partial y}{y \partial x} + \frac{C \partial\partial y}{y \partial x^2} + \frac{D \partial y \partial\partial y}{yy \partial x^3} + \frac{D y \partial^3 y - D \partial y \partial\partial y}{yy \partial x^3} = 0, \text{ seu}$$

$$Ay + B\frac{\partial y}{\partial x} + C\frac{\partial\partial y}{\partial x^2} + D\frac{\partial^3 y}{\partial x^3} = 0,$$

cujus resolutio est ostensa.

Scholion 3.

1124. Aequatio illa differentialis primi gradus

$$Dt\partial t + t\partial u(C + 3Du) + \partial u(A + Bu + Cu^2 + Du^3) = 0,$$

cujus integrale invenimus, diligentiore evolutione est digna. Ac primo quidem observo, eam integrabilem reddi, si dividatur per hanc formam

$$DDt^3 - Dtt(B + 2Cu + 3Duu) + t(C + 3Du)(A + Bu + Cu^2 + Du^3) + (A + Bu + Cu^2 + Du^3)^2,$$

unde concludimus et hanc aequationem

$$Dz\partial z + Duu\partial z + z\partial u(C + Du) + \partial u(A + Bu) = 0$$

integrabilem fieri, si dividatur per hanc formam

$$DDz^3 + Dzz(B + 2Cu) + z[AC + (3AD + BC)u + (BD + CC)uu] + AA + 2ABu + (AC + BB)uu + (AD + BC)u^3.$$

Utrinque autem divisor iste nihilo aequatus praebet integrale particulare, unde cum t vel z ternos obtineant valores, singuli exhibebunt integralia particularia. Hinc operae pretium erit in genere aequationem

$$y\partial y + yP\partial x + Q\partial x = 0,$$

investigare, quae per formam

$$y^3 + Lyy + My + N,$$

divisa integrabilis evadat. Per operationem autem supra explicatam invenitur

$$\partial L = 2P\partial x,\quad \partial M = PL\partial x + 3Q\partial x,$$
$$\partial N = 2QL\partial x,\ \text{et}\ PN - QM = 0;$$

unde colligitur

$$P\partial x = \tfrac{1}{2}\partial L,\quad Q\partial x = \frac{\partial N}{2L},$$
$$\partial M = \tfrac{1}{2}L\partial L + \frac{3\partial N}{2N},\ \text{et}\ N\partial L = \frac{M\partial N}{L},\ \text{seu}\ M = \frac{NL\partial L}{\partial N},$$

qui valor ibi substitutus sumto ∂N constante dat

$$3\partial N^2 = LL\partial L\partial N + 2NLL\partial\partial L + 2NL\partial L^2,$$

quae per ∂L multiplicata, transit in

$$3\partial L\partial N^2 = \partial . NL^2\partial L^2.$$

Verum commodius, ac singulari quidem modo, illae aequationes resolvuntur, statuendo

$$N = \alpha Z^2 \text{ et } L = \frac{\partial Z}{\partial z},$$

unde sumto elemento ∂z constante deducitur $M = \frac{Z\partial\partial Z}{2\partial z^2}$, hincque

$$\partial M = \frac{Z\partial^3 Z + \partial Z\partial\partial Z}{2\partial z^3} \text{ et}$$

$$\tfrac{1}{2}L\partial L + \frac{3\partial N}{2L} = \frac{\partial Z\partial\partial Z}{2\partial z^2} + 3\alpha Z\partial z.$$

Ergo $\partial^3 Z = 6\alpha\partial z^3$, ideoque

$$Z = \alpha z^3 + \beta zz + \gamma z + \delta,$$

$$P\partial x = \frac{\partial\partial Z}{2\partial z} \text{ et } Q\partial x = \alpha Z\partial z.$$

Quocirca sumto $Z = \alpha z^3 + \beta z^2 + \gamma z + \delta$, haec aequatio

$$y\partial y + y.\frac{\partial\partial Z}{2\partial z} + \alpha Z\partial z = 0$$

integrabilis redditur, divisa per hanc formam

$$y^3 + y^2.\frac{\partial Z}{\partial z} + y.\frac{Z\partial\partial Z}{2\partial z^2} + \alpha ZZ.$$

Praeterea si Z habeat factores, ut proposita sit haec aequatio

$$y\partial y + y\partial z(\alpha + \beta + \gamma + 3z) + \partial z(\alpha + z)(\beta + z)(\gamma + z) = 0$$

divisor eam integrabilem reddens erit

$$[y + (\alpha + z)(\beta + z)][y + (\alpha + z)(\gamma + z)][y + (\beta + z)(\gamma + z)],$$

cujus singuli factores nihilo aequati praebent integrale particulare. Ex unoquoque autem, more magis consueto, integrale completum ita elicitur. Ponatur

$$y = v - (\alpha + z)(\beta + z),$$

ac reperitur

$$v\partial v + v\partial z(\gamma + z) - \partial v(\alpha + z)(\beta + z) = 0,$$

sit porro $\partial v = p\partial z$, eritque $v = \frac{p(\alpha + z)(\beta + z)}{p + \gamma + z}$, et differentiando locoque ∂v ponendo $p\partial z$, orietur haec aequatio

$$\partial p(\alpha + z)(\beta + z)(\gamma + z) = \partial z[p^3 + (2\gamma - \alpha - \beta)p^2 + (\gamma - \alpha)(\gamma - \beta)p]$$

quae dat hanc separatam

$$\frac{\partial z}{(\alpha + z)(\beta + z)(\gamma + z)} = \frac{\partial p}{p(p + \gamma - \alpha)(p + \gamma - \beta)}.$$

Problema 145.

1125. Aequationis differentialis cujuscunque gradus

$$Ay + B\frac{\partial y}{\partial x} + C\frac{\partial\partial y}{\partial x^2} + D\frac{\partial^3 y}{\partial x^3} + E\frac{\partial^4 y}{\partial x^4} + \text{etc.} = 0$$

sumto elemento ∂x constante, integrale completum invenire.

Solutio.

Et huic aequationi evidens est satisfacere formulam $y = e^{\lambda x}$, cum enim hinc sit

$$\frac{\partial y}{\partial x} = \lambda e^{\lambda x}, \quad \frac{\partial\partial y}{\partial x^2} = \lambda^2 e^{\lambda x}, \quad \frac{\partial^3 y}{\partial x^3} = \lambda^3 e^{\lambda x},$$

et in genere $\frac{\partial^n y}{\partial x^n} = \lambda^n e^{\lambda x}$, facta substitutione pervenietur ad hanc aequationem, postquam scilicet per $e^{\lambda x}$ diviserimus,

$$A + B\lambda + C\lambda^2 + D\lambda^3 + E\lambda^4 + \text{etc.} = 0$$

ex qua valorem ipsius λ definiri oportet. Hinc littera λ totidem valores obtinebit, quoti fuerit ordinis aequatio differentialis proposita; quorum singuli aequationi aeque satisfaciunt. Qui valores si sint α, β, γ, δ, etc. integralia quidem particularia erunt

$$y = \mathfrak{A}e^{\alpha x}, \quad y = \mathfrak{B}e^{\beta x}, \quad y = \mathfrak{C}e^{\gamma x}, \text{ etc.}$$

Verum ex ipsa aequationis natura perspicuum est, aggregata quot-

cunque horum valorum, ideoque etiam omnium, perinde satisfacere. Cum igitur aggregatum omnium

$$y = \mathfrak{A} e^{\alpha x} + \mathfrak{B} e^{\beta x} + \mathfrak{C} e^{\gamma x} + \mathfrak{D} e^{\delta x} + \text{etc.}$$

tot complectatur constantes arbitrarias $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, etc. quoti ordinis differentialis est aequatio proposita, quin haec forma ejus sit integrale completum, dubitari nequit. Ascendat aequatio differentialis ad gradum n, ut sit

$$A y + B \frac{\partial y}{\partial x} + C \frac{\partial \partial y}{\partial x^2} + D \frac{\partial^3 y}{\partial x^3} + \ldots\ldots + N \frac{\partial^n y}{\partial x^n} = 0,$$

atque integrale completum ex n partibus constabit, quas ex resolutione hujus aequationis algebraicae ordinis n, scilicet

$$A + B\lambda + C\lambda^2 + D\lambda^3 + \ldots\ldots + N\lambda^n = 0;$$

definire oportet. Singuli nimirum ejus factores simplices partes illas patefacient, ita si factor sit $\alpha - \lambda$, ex eo integralis pars nascitur $\mathfrak{A} e^{\alpha x}$, quae, uti manifestum est, ex integratione aequationis differentialis simplicis $\alpha y - \frac{\partial y}{\partial x} = 0$ nascitur. Simili modo duo factores conjunctim

$$(\alpha - \lambda)(\beta - \lambda) = \alpha\beta - (\alpha + \beta)\lambda + \lambda\lambda$$

integralis portionem $\mathfrak{A} e^{\alpha x} + \mathfrak{B} e^{\beta x}$ suppeditant, quae simul est integrale hujus aequationis differentialis secundi gradus

$$\alpha\beta y - (\alpha + \beta)\frac{\partial y}{\partial x} + \frac{\partial \partial y}{\partial x^2} = 0.$$

Atque in genere si aequationis illius algebraicae factor sit

$$a + b\lambda + c\lambda^2 + f\lambda^3 + \text{etc.} = 0,$$

ex hoc vicissim formetur aequatio differentialis

$$a y + b\frac{\partial y}{\partial x} + c\frac{\partial \partial y}{\partial x^2} + f\frac{\partial^3 y}{\partial x^3} + \text{etc.} = 0,$$

cujus integrale completum si sit $y = P$, id simul erit pars integralis aequationis propositae. Atque hoc modo ex singulis factoribus aequationis algebraicae

$$A + B\lambda + C\lambda^2 + D\lambda^3 + \ldots\ldots + N\lambda^n = 0$$

derivabuntur singulae partes integralis quaesiti, quae junctae ejus integrale completum constituent, ita ut praecipuum negotium resolutioni hujus aequationis innitatur.

Corollarium 1.

1126. Si igitur istius aequationis algebraicae omnes factores simplices fuerint reales simulque inaequales, integratio nullam habet difficultatem. Si enim factor simplex sit $f + g\lambda$, integralis pars inde oriunda est $\mathfrak{A}e^{\frac{-fx}{g}}$.

Corollarium 2.

1127. Si bini factores simplices sint aequales, seu factor fuerit $(f + g\lambda)^2$, pars integralis inde oriunda est $e^{\frac{-fx}{g}}(\mathfrak{A} + \mathfrak{B}x)$. Si factor sit cubus $(f + g\lambda)^3$, inde oritur pars integralis

$$e^{\frac{-fx}{g}}(\mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2),$$

et ex factore biquadrato $(f + g\lambda)^4$ hujusmodi pars

$$e^{\frac{-fx}{g}}(\mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 + \mathfrak{D}x^3),$$

et ita porro pro quotcunque factoribus aequalibus, uti ex §. 1120. colligere licet.

Corollarium 3.

1128. Si factores occurrant imaginarii, bini conjuncti exhibent factorem trinomium realem, cujus forma ita repraesentatur

$$ff + 2fg\lambda\cos.\zeta + gg\lambda\lambda,$$

unde deducitur

$$\lambda = -\frac{f}{g}(\cos.\zeta \pm \sqrt{-1}.\sin.\zeta),$$

quo cum §. 1121. collato, fit $\mu = \frac{-f\cos.\zeta}{g}$ et $\nu = \frac{f\sin.\zeta}{g}$. Ex quo pars integralis ex tali factore oriunda erit

$$e^{\frac{-fx\cos.\zeta}{g}}\left(\mathfrak{A}\cos.\frac{fx\sin.\zeta}{g}+\mathfrak{B}\sin.\frac{fx\sin.\zeta}{g}\right).$$

Corollarium 4.

1129. Si hujusmodi formae quadratum inter factores occurrat

$$(ff+2fg\lambda\cos.\zeta+gg\lambda\lambda)^2,$$

seu duo hujusmodi factores sint aequales, considerentur quasi infinite parum discrepantes, ut in altero loco $\frac{f}{g}$ sit $\frac{f}{g}(1+\omega)$, et ob

$$e^{\frac{-fx\cos.\zeta}{g}(1+\omega)} = e^{\frac{-fx\cos.\zeta}{g}}\left(1-\frac{\omega fx}{g}\cos.\zeta\right),$$

$$\cos.\frac{fx\sin.\zeta}{g}(1+\omega) = \cos.\frac{fx\sin.\zeta}{g} - \frac{\omega fx\sin.\zeta}{g}\sin.\frac{fx\sin.\zeta}{g},\ \text{et}$$

$$\sin.\frac{fx\sin.\zeta}{g}(1+\omega) = \sin.\frac{fx\sin.\zeta}{g} + \frac{\omega fx\sin.\zeta}{g}\cos.\frac{fx\sin.\zeta}{g}:$$

ex hoc factore colligitur pars integralis

$$e^{\frac{-fx\cos.\zeta}{g}}\left\{\begin{array}{l}\mathfrak{A}'\cos.\frac{fx\sin.\zeta}{g}-\mathfrak{A}'\frac{\omega fx\cos.\zeta}{g}\cos.\frac{fx\sin.\zeta}{g}-\mathfrak{A}'\frac{\omega fx\sin.\zeta}{g}\sin.\frac{fx\sin.\zeta}{g}\\ +\mathfrak{B}'\sin.\frac{fx\sin.\zeta}{g}-\mathfrak{B}'\frac{\omega fx\cos.\zeta}{g}\sin.\frac{fx\sin.\zeta}{g}+\mathfrak{B}'\frac{\omega fx\sin.\zeta}{g}\cos.\frac{fx\sin.\zeta}{g}\end{array}\right\}$$

cui prior addi debet. Hunc in finem constantes ita contrahamus ponendo

$$\mathfrak{A}+\mathfrak{A}'=\mathfrak{E},\quad \frac{-\mathfrak{A}'\omega f\cos.\zeta}{g}+\frac{\mathfrak{B}'\omega f\sin.\zeta}{g}=\mathfrak{G},$$

$$\mathfrak{B}+\mathfrak{B}'=\mathfrak{F},\quad \frac{-\mathfrak{A}'\omega f\sin.\zeta}{g}-\frac{\mathfrak{B}'\omega f\cos.\zeta}{g}=\mathfrak{H},$$

unde illae constantes utique determinantur, eritque pars integralis respondens

$$e^{\frac{-fx\cos.\zeta}{g}}\left[(\mathfrak{E}+\mathfrak{G}x)\cos.\frac{fx\sin.\zeta}{g}+(\mathfrak{F}+\mathfrak{H}x)\sin.\frac{fx\sin.\zeta}{g}\right],$$

Scholion.

1130. En ergo universam methodum hujusmodi aequationum differentialium integralia inveniendi

$$Ay + B\frac{\partial y}{\partial x} + C\frac{\partial\partial y}{\partial x^2} + D\frac{\partial^3 y}{\partial x^3} + \ldots + N\frac{\partial^n y}{\partial x^n} = 0$$

ita in compendium contractam. Procedatur ut iste laterculus indicat

loco	y	$\frac{\partial y}{\partial x}$	$\frac{\partial\partial y}{\partial x^2}$	$\frac{\partial^3 y}{\partial x^3}$	$\frac{\partial^4 y}{\partial x^4}$	..... $\frac{\partial^n y}{\partial x^n}$
scribatur	1	z	z^2	z^3	z^4	..... z^n

u oriatur haec aequatio algebraica

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + \ldots + Nz^n = 0,$$

cujus singuli factores reales, sive simplices sive duplicati, notentur, atque insuper casus quibus duo pluresve sunt inter se aequales, probe observentur. Tum cujusmodi partes pro integrali quaesito ex singulis factoribus oriantur, ex sequente tabella intelligere licet:

Factores	Partes integralis
$f + gz$	$\mathfrak{A}\, e^{\frac{-fx}{g}}$
$(f + gz)^2$	$(\mathfrak{A} + \mathfrak{B}x)\, e^{\frac{-fx}{g}}$
$(f + gz)^3$	$(\mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2)\, e^{\frac{-fx}{g}}$
$(f + gz)^4$	$(\mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 + \mathfrak{D}x^3)\, e^{\frac{-fx}{g}}$
$ff + 2fgz\cos.\zeta + ggzz$	$e^{\frac{-fx\cos.\zeta}{g}}\left(\mathfrak{A}\cos.\frac{fx\sin.\zeta}{g} + \mathfrak{B}\sin.\frac{fx\sin.\zeta}{g}\right)$
$(ff + 2fgz\cos.\zeta + ggzz)^2$	$e^{\frac{-fx\cos.\zeta}{g}}\left\{\begin{array}{l}(\mathfrak{A} + \mathfrak{B}x)\cos.\frac{fx\sin.\zeta}{g}\\ (\mathfrak{a} + \mathfrak{b}x)\sin.\frac{fx\sin.\zeta}{g}\end{array}\right\}$
$(ff + 2fgz\cos.\zeta + ggzz)^3$	$e^{\frac{-fx\cos.\zeta}{g}}\left\{\begin{array}{l}(\mathfrak{A} + \mathfrak{B}x + \mathfrak{C}xx)\cos.\frac{fx\sin.\zeta}{g}\\ (\mathfrak{a} + \mathfrak{b}x + \mathfrak{c}xx)\sin.\frac{fx\sin.\zeta}{g}\end{array}\right\}$

etc.

Pro singulis autem factoribus diversae litterae constantes scribi debent, ut integrale omnibus numeris completum obtineatur.

Exemplum 1.

1131. *Aequationis differentialis quarti gradus*

$$y - \frac{2\partial y}{\partial x} + \frac{2\partial\partial y}{\partial x^2} - \frac{2\partial^3 y}{\partial x^3} + \frac{2\partial^4 y}{\partial x^4} = 0$$

integrale completum assignare.

Hinc oritur aequatio algebraica

$$1 - 2z + 2zz - 2z^3 + z^4 = 0.$$

quae in hos factores resolvitur $(1-z)^2(1+zz)$, quorum prior ob $f=1$ et $g=-1$ praebet hanc partem integralis $(\mathfrak{A}+\mathfrak{B}x)e^x$, alter vero factor ob $f=1$, $\cos.\zeta=0$, $g=1$, et $\sin.\zeta=1$, dat $\mathfrak{A}\cos.x+\mathfrak{B}\sin.x$. Quare integrale completum, quod quaeritur, erit

$$y = (\mathfrak{A}+\mathfrak{B}x)e^x + \mathfrak{C}\cos.x + \mathfrak{D}\sin.x$$

continens quatuor constantes arbitrarias. Quod si velimus, ut posito $x=0$ fiat $y=0$, fieri oportet $\mathfrak{A}+\mathfrak{C}=0$, si etiam $\frac{\partial y}{\partial x}$ eodem casu evanescere debeat, ob

$$\frac{\partial y}{\partial x} = (\mathfrak{A}+\mathfrak{B}+\mathfrak{B}x)e^x - \mathfrak{C}\sin.x + \mathfrak{D}\cos.x$$

fieri debet $\mathfrak{A}+\mathfrak{B}+\mathfrak{D}=0$. Si praeterea $\frac{\partial\partial y}{\partial x^2}$ evanescere debeat, ob

$$\frac{\partial\partial y}{\partial x^2} = (\mathfrak{A}+2\mathfrak{B}+\mathfrak{B}x)e^x - \mathfrak{C}\cos.x - \mathfrak{D}\sin.x$$

fieri debet $\mathfrak{A}+2\mathfrak{B}-\mathfrak{C}=0$. Quare his tribus conditionibus satisfaciemus sumendo $\mathfrak{C}=-\mathfrak{A}$, $\mathfrak{B}=-\mathfrak{A}$ et $\mathfrak{D}=0$, ita ut sit integrale

$$y = \mathfrak{A}(1-x)e^x - \mathfrak{A}\cos.x.$$

Exemplum 2.

1132. *Aequationem differentialem quarti ordinis*

$$Ay + C\frac{\partial\partial y}{\partial x^2} + E\frac{\partial^4 y}{\partial x^4} = 0,$$

sumto elemento ∂x constante, integrare.

Aequatio algebraica ad integrationem perducens est

$$A + Czz + Ez^4 = 0,$$

quae semper duos factores duplicatos reales habet, quorum forma duplex esse potest

vel $(aa + 2maz + nzz)(aa - 2maz + nzz)$
vel $(aa + mzz)(aa + nzz)$.

Ex priori est

$$A = a^4,\ C = 2naa - 4mmaa,\ E = nn,$$

ex posteriore vero

$$A = a^4,\ C = (m+n)aa,\ E = mn:$$

semper autem terminum primum A biquadrato a^4 repraesentare licet, et prior resolutio locum habet, si E sit numerus positivus, et $2naa - C$ seu $2\sqrt{AE} - C$ quoque positivus, ideoque $4AE > CC$. Posterior vero si $CC > 4AE$. Tum igitur videndum est, ad quamnam classem singuli factores pertineant, unde sequentes casus occurrent.

I. Si omnes quatuor factores simplices sunt reales, erit

$$A + Czz + Ez^4 = (a+z)(a-z)(b+z)b-z),$$

et haec habebitur aequatio

$$aabby - (aa+bb)\frac{\partial\partial y}{\partial x^2} + \frac{\partial^4 y}{\partial x^4} = 0,$$

cujus integrale completum est

$$y = \mathfrak{A}e^{ax} + \mathfrak{B}e^{-ax} + \mathfrak{C}e^{bx} + \mathfrak{D}e^{-bx}.$$

Ac si sit $b = a$, hujus aequationis

$$a^4 y - \frac{2aa\partial\partial y}{\partial x^2} + \frac{\partial^4 y}{\partial x^4} = 0,$$

integrale completum erit

$$y = (\mathfrak{A} + \mathfrak{B} x) e^{ax} + (\mathfrak{C} + \mathfrak{D} x) e^{-ax}.$$

II. Si duo factores simplices sint reales, duo vero imaginarii, erit

$$A + Czz + Ez^4 = (a + z)(a - z)(bb + zz),$$

et haec habebitur aequatio

$$aabby + (aa - bb)\frac{\partial\partial y}{\partial x^2} + \frac{\partial^4 y}{\partial x^4} = 0,$$

cujus integrale completum est

$$y = \mathfrak{A} e^{ax} + \mathfrak{B} e^{-ax} + \mathfrak{C} \cos. bx + \mathfrak{D} \sin. bx.$$

III. Si omnes factores simplices sint imaginarii, duo casus sunt evolvendi:

1) si $A + Czz + Ez^4 = (aa + zz)(bb + zz)$,

unde hujus aequationis

$$aabby + (aa + bb)\frac{\partial\partial y}{\partial x^2} + \frac{\partial^4 y}{\partial x^4} = 0$$

integrale completum erit

$$y = \mathfrak{A} \cos. ax + \mathfrak{B} \sin. ax + \mathfrak{C} \cos. bx + \mathfrak{D} \sin. bx:$$

2) si $A + Czz + Ez^4 = (aa + 2az\cos.\zeta + zz)(aa - 2az\cos.\zeta + zz)$,

unde hujus aequationis

$$a^4 y - \frac{2aa\partial\partial y}{\partial x^2}\cos. 2\zeta + \frac{\partial^4 y}{\partial x^4} = 0,$$

integrale completum est

$$y = e^{+ax\cos.\zeta}(\mathfrak{A} \cos. ax \sin. \zeta + \mathfrak{B} \sin. ax \sin. \zeta)$$
$$+ e^{-ax\cos.\zeta}(\mathfrak{C} \cos. ax \sin. \zeta + \mathfrak{D} \sin. ax \sin. \zeta).$$

At si sit priori casu $b = a$, seu posteriori $\cos. \zeta = 0$, hujus aequationis

$$a^4 y + \frac{2aa\partial\partial y}{\partial x^2} + \frac{\partial^4 y}{\partial x^4} = 0$$

integrale completum est

$$y = (\mathfrak{A} + \mathfrak{B} x) \cos. a x + (\mathfrak{C} + \mathfrak{D} x) \sin. a x.$$

Scholion 1.

1133. Cum igitur aequationis

$$A y + \frac{C\partial\partial y}{\partial x^2} + \frac{E\partial^4 y}{\partial x^4} = 0$$

integrale assignari possit, omnes aequationes quas inde derivare licet, integrari poterunt. At haec aequatio per $2\,\partial y$ multiplicata per integrationem ad differentialem tertii ordinis reducitur

$$A y y + \frac{C\partial y^2}{\partial x^2} + \frac{2E\partial y \partial^3 y - E\partial\partial y^2}{\partial x^4} = \text{Const.}$$

In integrali autem ante invento constantes $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ ita definire licet, ut haec Const. evanescat, ideoque hujus aequationis

$$A y y + \frac{C\partial y^2}{\partial x^2} + \frac{2E\partial y \partial^3 y - E\partial\partial y^2}{\partial x^4} = 0$$

integrale completum erit in nostra potestate. Nunc ponatur $y = e^{\int v \partial x}$, ut sit $v = \frac{\partial y}{y \partial x}$, et ob $\frac{\partial y}{\partial x} = e^{\int v \partial x} v$,

$$\frac{\partial\partial y}{\partial x^2} = e^{\int x \partial x}\left(\frac{\partial v}{\partial x} + v v\right), \text{ atque}$$

$$\frac{\partial^3 y}{\partial x^3} = e^{\int v \partial x}\left(\frac{\partial\partial v}{\partial x^2} + \frac{3 v \partial v}{\partial x} + v^3\right),$$

aequatio nostra hanc induit formam

$$A + C v v + E\left(\frac{2 v \partial\partial v}{\partial x^2} + \frac{4 v v \partial v}{\partial x} + v^4 - \frac{\partial v^2}{\partial x^2}\right) = 0.$$

Sit porro $\partial x = \frac{\partial v}{s}$, ut sit

$$s = \frac{\partial v}{\partial x} = \frac{\partial\partial y}{y \partial x^2} - \frac{\partial y^2}{y y \partial x^2}, \text{ erit}$$

$$\frac{\partial\partial v}{\partial x} = \partial s \text{ et } \frac{\partial\partial v}{\partial x^2} = \frac{s \partial s}{\partial v}:$$

unde resultat haec aequatio differentialis primi gradus

$$A + C x v + E\left(\frac{2 v s \partial s}{\partial v} - s s + 4 v v s + v^4\right) = 0,$$

cujus relatio inter v et s ita ex relatione inter x et y inventa definitur, ut sit

$$v = \frac{\partial y}{y \partial x} \text{ et } s = \frac{y \partial\partial y - \partial y^2}{y y \partial x^2}.$$

Scholion 2.

1134. Retenta autem illa constante per integrationem ingressa, ut habeatur

$$A y y + \frac{C \partial y^2}{\partial x^2} + \frac{2 E \partial y \partial^3 y - E \partial\partial y^2}{\partial x^4} = G,$$

in integrali completo, quo y per x exprimitur, constantes $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, quantitati huic G conformiter determinari poterunt. Nunc igitur ponatur $\partial x = \frac{\partial y}{u}$, ut sit $\frac{\partial y}{\partial x} = u$, erit

$$\frac{\partial\partial y}{\partial x^2} = \frac{u \partial u}{\partial y}, \text{ et } \frac{\partial^3 y}{\partial x^2} = \partial . \frac{u \partial u}{\partial y},$$

ideoque $\frac{\partial^3 y}{\partial x^3} = \frac{u}{\partial y} \partial . \frac{u \partial u}{\partial y}$. Unde obtinetur haec aequatio differentialis secundi gradus

$$A y y + C u u + E \left(\frac{2 u u}{\partial y} \partial . \frac{u \partial u}{\partial y} - \frac{u u \partial u^2}{\partial y^2} \right) = G,$$

ubi consideratio elementi pro constante assumti est exuta. Nihil ergo impedit, quo minus sumamus ∂y pro constante, fietque

$$A y y + C u u + E \left(\frac{2 u^3 \partial\partial u}{\partial y^2} + \frac{u u \partial u^2}{\partial y^2} \right) = G,$$

quae ergo aequatio etiam integrari potest.

Vel si ponamus $y y = p$ et $u u = q$, sumto elemento ∂p constante, prodibit haec aequatio

$$A p + C q + E \left(\frac{4 p q \partial\partial q + 2 q \partial p \partial q - p \partial q^2}{\partial p^2} \right) = G.$$

Vel si in illa aequatione ponatur $u = r^{\frac{2}{3}}$, erit

$$A y y + C r^{\frac{4}{3}} + \tfrac{4}{3} E r^{\frac{5}{3}} . \frac{\partial\partial r}{\partial y^2} = G.$$

Quarum formarum integratio sine hoc subsidio maxime ardua videtur.

Problema 146.

1135. Proposita aequatione differentiali ordinis cujuscunque $a^n y \pm \frac{\partial^n y}{\partial x^n} = 0$, ubi elementum ∂x constans est assumtum, ejus integrale completum investigare.

Solutio.

Aequatio algebraica solutioni inserviens est $a^n \pm z^n = 0$, pro cujus resolutione duos casus considerari convenit, prout signum vel superius vel inferius valeat.

I. Valeat superius, ut haec proposita sit aequatio

$$a^n y + \frac{\partial^n y}{\partial x^n} = 0,$$

et formulae $a^n + z^n$ factores reales sunt

$$aa - 2az\cos.\frac{\pi}{n} + zz,\quad aa - 2az\cos.\frac{3\pi}{n} + zz,$$
$$aa - 2az\cos.\frac{5\pi}{n} + zz,\ \text{etc.}$$

quorum ultimus est vel $aa - 2az\cos.\frac{n\pi}{n} + zz$ vel

$$aa - 2az\cos.\frac{(n-1)\pi}{n} + zz,$$

prout vel n vel $n-1$ fuerit numerus impar, atque illo quidem casu loco factoris quadrati $aa + 2az + zz$ ejus radix $a + z$ sumi debet.

Hinc istius aequationis integrale completum est

$$y = + e^{ax\cos.\frac{\pi}{n}} (\mathfrak{A}\cos.ax\sin.\tfrac{\pi}{n} + \mathfrak{B}\sin.ax\sin.\tfrac{\pi}{n})$$
$$+ e^{ax\cos.\frac{3\pi}{n}} (\mathfrak{C}\cos.ax\sin.\tfrac{3\pi}{n} + \mathfrak{D}\sin.ax\sin.\tfrac{3\pi}{n})$$
$$+ e^{ax\cos.\frac{5\pi}{n}} (\mathfrak{E}\cos.ax\sin.\tfrac{5\pi}{n} + \mathfrak{F}\sin.ax\sin.\tfrac{5\pi}{n})$$
$$\text{etc.}$$

cujus expressionis, si n sit numerus impar, ultima pars fit $\mathfrak{N}\,e^{-ax}$. Quod integrale etiam ita potest exhiberi

$$y = \mathfrak{A}\,e^{ax\cos.\frac{\pi}{n}}\cos.\left(ax\sin.\frac{\pi}{n}+\mathfrak{a}\right) + \mathfrak{B}\,e^{ax\cos.\frac{3\pi}{n}}\cos.\left(ax\sin.\frac{3\pi}{n}+\mathfrak{b}\right)$$

$$+\mathfrak{C}\,e^{ax\cos.\frac{5\pi}{n}}\cos.\left(ax\sin.\frac{5\pi}{n}+\mathfrak{c}\right)+\mathfrak{D}\,e^{ax\cos.\frac{7\pi}{n}}\cos.\left(ax\sin.\frac{7\pi}{n}+\mathfrak{d}\right)$$

etc.

quae forma eousque continuari debet, quoad similes termini recurrant.

II. Si valeat signum inferius, propositaque sit haec aequatio

$$a^n y - \frac{\partial^n y}{\partial x^n} = 0,$$

formulae $a^n - z^n$ factores reales sunt

$$a - z,\quad aa - 2z\cos.\frac{2\pi}{n} + zz,\quad aa - 2az\cos.\frac{4\pi}{n} + zz,$$

$$aa - 2az\cos.\frac{6\pi}{n} + zz,\ \text{etc.}$$

quorum si n numerus par, ultimus est $a + z$, sin autem impar

$$aa - 2az\cos.\frac{(n-1)\pi}{n} + zz.$$

Quare aequationis hujus integrale completum est

$$y = \mathfrak{A}\,e^{ax} + e^{ax\cos.\frac{2\pi}{n}}\left(\mathfrak{B}\cos.ax\sin.\frac{2\pi}{n} + \mathfrak{C}\sin.ax\sin.\frac{2\pi}{n}\right)$$

$$+ e^{ax\cos.\frac{4\pi}{n}}\left(\mathfrak{D}\cos.ax\sin.\frac{4\pi}{n} + \mathfrak{E}\sin.ax\sin.\frac{4\pi}{n}\right)$$

$$+ e^{ax\cos.\frac{6\pi}{n}}\left(\mathfrak{F}\cos.ax\sin.\frac{6\pi}{n} + \mathfrak{G}\sin.ax\sin.\frac{6\pi}{n}\right),$$

etc.

quod integrale etiam ita exprimi potest

$$y = \mathfrak{A}\,e^{ax} + \mathfrak{B}\,e^{ax\cos.\frac{2\pi}{n}}\cos.\left(ax\sin.\frac{2\pi}{n} + \mathfrak{b}\right)$$

$$+\mathfrak{C}\,e^{ax\cos.\frac{4\pi}{n}}\cos.\left(ax\sin.\frac{4\pi}{n}+\mathfrak{c}\right)+\mathfrak{D}\,e^{ax\cos.\frac{6\pi}{n}}\cos.\left(ax\sin.\frac{6\pi}{n}+\mathfrak{d}\right),$$

etc.

quae forma eousque est continuanda, quamdiu termini a prioribus diversi prodeunt.

Scholion 1.

1136. Pro variis ergo exponentis n valoribus integralia sequenti modo se habebunt, ac primo quidem pro aequatione $a^n y + \frac{\partial^n y}{\partial x^n} = 0$.

I. Aequationis $a y + \frac{\partial y}{\partial x} = 0$ integrale est

$$y = \mathfrak{A}\, e^{-ax}.$$

II. Aequationis $a^2 y + \frac{\partial\partial y}{\partial x^2} = 0$ integrale est

$$y = \mathfrak{A} \cos. (a x + \mathfrak{a}).$$

III. Aequationis $a^3 y + \frac{\partial^3 y}{\partial x^3} = 0$ integrale est

$$y = \mathfrak{A}\, e^{\frac{1}{2} a x} \cos. \left(\frac{a x \sqrt{3}}{2} + \mathfrak{a}\right) + \mathfrak{B}\, e^{-ax}.$$

IV. Aequationis $a^4 y + \frac{\partial^4 y}{\partial x^4} = 0$ integrale est

$$y = \mathfrak{A}\, e^{\frac{a x}{\sqrt{2}}} \cos. \left(\frac{a x}{\sqrt{2}} + \mathfrak{a}\right) + \mathfrak{B}\, e^{\frac{-a x}{\sqrt{2}}} \cos. \left(\frac{a x}{\sqrt{2}} + \mathfrak{b}\right).$$

V. Aequationis $a^5 y + \frac{\partial^5 y}{\partial x^5} = 0$ integrale est

$$y = \mathfrak{A}\, e^{a x \cos. 36^0} \cos. (a x \sin. 36^0 + \mathfrak{a})$$
$$+ \mathfrak{B}\, e^{-a x \cos. 72^0} \cos. (a x \sin. 72^0 + \mathfrak{b}) + \mathfrak{C}\, e^{-ax}.$$

VI. Aequationis $a^6 y + \frac{\partial^6 y}{\partial x^6} = 0$ integrale est

$$y = \mathfrak{A}\, e^{\frac{a x \sqrt{3}}{2}} \cos. (\tfrac{1}{2} a x + \mathfrak{a})$$
$$+ \mathfrak{B} \cos. (a x + \mathfrak{b}) + \mathfrak{C}\, e^{\frac{-a x \sqrt{3}}{2}} \cos. (\tfrac{1}{2} a x + \mathfrak{c}),$$

etc.

Simili autem modo pro altera forma

$$a^n y - \frac{\partial^n y}{\partial x^n} = 0,$$

integrationes ad valores simpliciores exponentis n accommodatae ita se habebunt

I. Aequationis $a y - \frac{\partial y}{\partial x} = 0$ integrale est
$y = \mathfrak{A} e^{ax}$.

II. Aequationis $a^2 y - \frac{\partial \partial y}{\partial x^2} = 0$ integrale est
$y = \mathfrak{A} e^{ax} + \mathfrak{B} e^{-ax}$.

III. Aequationis $a^3 y - \frac{\partial^3 y}{\partial x^3} = 0$ integrale est

$$y = \mathfrak{A} e^{ax} + \mathfrak{B} e^{-\frac{1}{2} a x} \cos. \left(\frac{a x \sqrt{3}}{2} + \mathfrak{b}\right).$$

IV. Aequationis $a^4 y - \frac{\partial^4 y}{\partial x^4} = 0$ integrale est
$y = \mathfrak{A} e^{ax} + \mathfrak{B} \cos. (a x + \mathfrak{b}) + \mathfrak{C} e^{-ax}$.

V. Aequationis $a^5 y - \frac{\partial^5 y}{\partial x^5} = 0$ integrale est

$$y = \mathfrak{A} e^{ax} + \mathfrak{B} e^{a x \cos. 72^0} \cos. (a x \sin. 72^0 + \mathfrak{b})$$
$$+ \mathfrak{C} e^{-a x \cos. 36^0} \cos. (a x \sin. 36^0 + \mathfrak{c}).$$

VI. Aequationis $a^6 y - \frac{\partial^6 y}{\partial x^6} = 0$ integrale est

$$y = \mathfrak{A} e^{ax} + \mathfrak{B} e^{\frac{1}{2} a x} \cos. \left(\frac{a x \sqrt{3}}{2} + \mathfrak{b}\right)$$
$$+ \mathfrak{C} e^{-\frac{1}{2} a x} \cos. \left(\frac{a x \sqrt{3}}{2} + \mathfrak{c}\right) + \mathfrak{D} e^{-ax},$$

sicque quousque libuerit progredi licet

Scholion 2.

1137. Quamvis methodus, qua hic sum usus, expedite integralia aequationum in proposita forma contentarum suppeditet, a

principiis tamen integrationis omnino abhorret. Cum enim aequatio differentialis est altioris gradus, leges integrationis postulant, ut toties seorsim integretur, antequam ad relationem finitam inter binas variabiles perveniatur, et dum singulae integrationes constantem arbitrariam recipiunt, hoc demum modo integrale completum obtinetur. Hactenus autem una quasi operatione integrale postremum eruimus, cum omnibus constantibus, quibus id completum redditur; revera scilicet sola conjectura utentes plura integralia particularia sumus adepti, atque natura aequationis commode permisit, ut ex iis integrale completum formare liceret. Verum si leges integrandi stricte observare velimus, proposita verbi gratia aequatione differentiali quarti gradus, quadruplici integratione opus erit, quarum prima ea reducatur ad aequationem differentialem tertii gradus, tum vero ista per novam integrationem ad aequationem differentialem secundi gradus, quae denuo integrata ad gradum primum perducatur, haecque tandem iterum integrata relationem quaesitam inter binas variabiles patefaciat. Atque hoc modo etiam aequationum hic tractatarum formam resolvere licet, ut per continuas integrationes ad gradus simpliciores redigatur, quibus tandem eadem integralia, quae hic elicuimus, inveniantur. Cum autem haec methodus latius pateat, quam ad formas hic consideratas, ejusque ope haec aequatio generalior integrari queat

$$X = Ay + \frac{B\partial y}{\partial x} + \frac{C\partial\partial y}{\partial x^2} + \frac{D\partial^3 y}{\partial x^3} + \text{etc.}$$

denotante X functionem quamcunque ipsius x, cui resolvendae praecedens methodus minime sufficit, novam methodum statim ad hanc formam generaliorem accommodabo.

CAPUT III.

DE

INTEGRATIONE AEQUATIONUM DIFFERENTIALIUM HUJUS FORMAE

$$X = Ay + \frac{B\partial y}{\partial x} + \frac{C\partial\partial y}{\partial x^2} + \frac{D\partial^3 y}{\partial x^3} + \text{etc.}$$

Problema 147.

1138.

Proposita aequatione differentiali

$$X = Ay + \frac{B\partial y}{\partial x} + \frac{C\partial\partial y}{\partial x^2} + \frac{D\partial^3 y}{\partial x^3} + \ldots + \frac{N\partial^n y}{\partial x^n},$$

sumto elemento ∂x constante, et significante X functionem quamcunque ipsius x, invenire functionem ipsius x, per quam haec aequatio multiplicata integrabilis evadat.

Solutio.

Sit $P\partial x$ iste multiplicator quem quaerimus, et cum prius membrum X eo integrabile reddatur, ejus rationem ex altero membro definiri oportet. Facile autem intelligitur, formam hujus multiplicatoris P ejusmodi fore $e^{\lambda x}$, ita ut quantitas λ definiri debeat. Sit ergo $e^{\lambda x}\partial x$ multiplicator, atque hanc formam

$$e^{\lambda}\partial x\left(Ay + \frac{B\partial y}{\partial x} + \frac{C\partial\partial y}{\partial x^2} + \frac{D\partial^3 y}{\partial x^3} + \ldots + \frac{N\partial^n y}{\partial x^n}\right),$$

integrabilem esse oportet, cujus integrale propterea statuatur

$$e^{\lambda x}\left(A'y + \frac{B'\partial y}{\partial x} + \frac{C'\partial\partial y}{\partial x^2} + \ldots\ldots + \frac{M'\partial^{n-1}y}{\partial x^{n-1}}\right),$$

ita ut hujus differentiale cum illa forma congruere debet, quod cum sit

$$e^{\lambda x}\partial x\left\{\begin{array}{l}\lambda A'y + \frac{\lambda B'\partial y}{\partial x} + \frac{\lambda C'\partial\partial y}{\partial x^2} + \ldots\ldots + \frac{\lambda M'\partial^{n-1}y}{\partial x^n} \\ + \frac{A'\partial y}{\partial x} + \frac{B'\partial\partial y}{\partial x^2} + \ldots\ldots\ldots\ldots\ldots + \frac{M'\partial^n y}{\partial x^n}\end{array}\right\},$$

necesse est sit

$$A' = \frac{A}{\lambda},\ B' = \frac{B - A'}{\lambda},\ C' = \frac{C - B'}{\lambda},\ D' = \frac{D - C'}{\lambda},\ \ldots\ldots$$

$\ldots\ldots M' = \frac{M - L'}{\lambda}$, atque $M' = N$. Hinc erit

$$A' = \frac{A}{\lambda},$$
$$B' = \frac{B}{\lambda} - \frac{A}{\lambda^2},$$
$$C' = \frac{C}{\lambda} - \frac{B}{\lambda^2} + \frac{A}{\lambda^3},$$
$$D' = \frac{D}{\lambda} - \frac{C}{\lambda^2} + \frac{B}{\lambda^3} - \frac{A}{\lambda^4},$$

$$\vdots$$

$$M' = \frac{M}{\lambda} - \frac{L}{\lambda^2} + \frac{K}{\lambda^3} \ldots\ldots \pm \frac{A}{\lambda^n}, \text{ et}$$

$$0 = \frac{N}{\lambda} - \frac{M}{\lambda^2} + \frac{L}{\lambda^3} \ldots\ldots \mp \frac{A}{\lambda^{n+1}},$$

ubi ex ultima aequatione quantitas λ erui debet, quae aequatio induit hanc formam

$$A - B\lambda + C\lambda^2 - D\lambda^3 + E\lambda^4 \ldots\ldots \pm N\lambda^n = 0$$

unde cum λ sortiatur n valores, totidem quoque multiplicatores, inveniuntur.

Videamus, quomodo hae determinationes pro singulis valoribus exponentis n se habeant.

I. Si $n = 1$; erit $A - B\lambda = 0$, tum vero

$$A' = \frac{A}{\lambda} = B.$$

II. Si $n = 2$; erit $A - B\lambda + C\lambda^2 = 0$, tum vero

$$A' = \frac{A}{\lambda} = B - C\lambda \text{ et } B' = \frac{B\lambda - A}{\lambda^2} = C.$$

III. Si $n = 3$; erit $A - B\lambda + C\lambda^2 - D\lambda^3 = 0$ tum vero

$$A' = \frac{A}{\lambda} = B - C\lambda + D\lambda^2,$$

$$B' = \frac{B\lambda - A}{\lambda^2} = C - D\lambda \text{ et}$$

$$C' = \frac{C\lambda^2 - B\lambda + A}{\lambda^3} = D.$$

IV. Si $n = 4$; erit $A - B\lambda + C\lambda^2 - D\lambda^3 + E\lambda^4 = 0$, tum vero

$$A' = \frac{A}{\lambda} = B - C\lambda + D\lambda^2 - E\lambda^3,$$

$$B' = \frac{B\lambda - A}{\lambda^2} = C - D\lambda + E\lambda^2,$$

$$C' = \frac{C\lambda^2 - B\lambda + A}{\lambda^3} = D - E\lambda,$$

$$D' = \frac{D\lambda^3 - C\lambda^2 + B\lambda - A}{\lambda^4} = E.$$

V. Si $n = 5$; erit $A - B\lambda + C\lambda^2 - D\lambda^3 + E\lambda^4 - F\lambda^5 = 0$, tum vero

$$A' = \frac{A}{\lambda} = B - C\lambda + D\lambda^2 - E\lambda^3 + F\lambda^4$$

$$B' = \frac{B\lambda - A}{\lambda^2} = C - D\lambda + E\lambda^2 - F\lambda^3$$

$$C' = \frac{C\lambda^2 - B\lambda + A}{\lambda^3} = D - E\lambda + F\lambda^2$$

$$D' = \frac{D\lambda^3 - C\lambda^2 + B\lambda - A}{\lambda^4} = E - F\lambda$$

$$E' = \frac{E\lambda^4 - D\lambda^3 + C\lambda^2 - B\lambda + A}{\lambda^5} = F$$

atque ita porro.

Invento autem hoc multiplicatore $e^{\lambda x}\partial x$, prius aequationis membrum fit $\int e^{\lambda x}\mathrm{X}\,\partial x$, et aequatio proposita, quae est differentialis gradus n, per integrationem reducitur ad hanc uno gradu simpliciorem

$$\int e^{\lambda x}\mathrm{X}\,\partial x = e^{\lambda x}\left(\mathrm{A}'y + \mathrm{B}'\frac{\partial y}{\partial x} + \mathrm{C}'\frac{\partial\partial y}{\partial x^2} + \ldots\ldots + \mathrm{M}'\frac{\partial^{n-1}y}{\partial x^{n-1}}\right).$$

Corollarium 1.

1139. Intagratione ergo hac prima instituta, aequatio proposita uno gradu deprimitur, et definitis coëfficientibus A′, B′, C′, etc. ex superioribus formulis, aequatio integralis hac forma exhiberi potest

$$e^{-\lambda x}\int e^{\lambda x}\mathrm{X}\,\partial x = \mathrm{A}'y + \mathrm{B}'\frac{\partial y}{\partial x} + \mathrm{C}'\frac{\partial\partial y}{\partial x^2} + \ldots\ldots + \mathrm{M}'\frac{\partial^{n-1}y}{\partial x^{n-1}}.$$

Corollarium 2.

1140. Cum prius membrum $e^{-\lambda x}\int e^{\lambda x}\mathrm{X}\,\partial x$ sit functio ipsius x constantem arbitrariam involvens, si ejus loco ponatur X′, haec aequatio similem formam habet atque ipsa proposita, ideoque eadem methodo iterum integrari et ad gradum differentialitatis $n-2$ reduci potest, quae hujusmodi formam habebit

$$\mathrm{X}'' = \mathrm{A}''y + \mathrm{B}''\frac{\partial y}{\partial x} + \mathrm{C}''\frac{\partial\partial y}{\partial x^2} + \ldots\ldots\ldots + \mathrm{L}''\frac{\partial^{n-2}y}{\partial x^{n-2}}.$$

Corollarium 3.

1141. Hoc modo ulterius progrediendo tandem ad aequationem differentialem primi gradus pervenietur

$$\mathrm{X}^{(n-1)} = \mathrm{A}^{(n-1)}y + \mathrm{B}^{(n-1)}\frac{\partial y}{\partial x},$$

quae simili modo ad aequationem finitam $\mathrm{X}^{(n)} = \mathrm{A}^{(n)}y$ reducitur, qua relatio inter ipsas variabiles x et y exprimitur.

Scholion.

1142. Haec igitur est methodus hujusmodi aequationes differentiales altiorum graduum successive per gradus integrandi, ubi tot opus est integrationibus, quoti gradus differentialis fuerit ipsa aequatio proposita. Totum ergo negotium situm est in inventione successiva coëfficientium, quos ex praecedentibus ope multiplicatoris definiri oportet. In genere quidem lex, qua ii continuo ex antecedentibus determinantur, non ita est perspicua, ut inde forma integrabilis extremi perspici possit; verum quia ex capite superiori novimus, casu quo primum membrum X evanescit, etiam ultimum integrale lege satis simplici contineri, idem hic usu venire merito suspicamur, eamque legem facillime agnoscemus, si pedetentim a gradibus inferioribus ad altiores progrediamur. Ac primo quidem casu, quo aequatio est differentialis primi gradus $X = Ay + B\frac{\partial y}{\partial x}$, multiplicator erit $e^{\lambda x}\partial x$, posito $A - \lambda B = 0$, ut sit $\lambda = \frac{A}{B}$, et cum sit $A' = \frac{A}{\lambda} = B$, integrale erit

$$\int e^{\lambda x} X \partial x = B e^{\lambda x} y \text{ seu } e^{-\lambda x}\int e^{\lambda x} X \partial x = By.$$

Ad hanc similitudinem aequationes graduum altiorum evolvamus, ac formam integralis ultimi investigemus.

Problema 148.

1143. Proposita aequatione differentiali secundi gradus

$$X = Ay + B\frac{\partial y}{\partial x} + C\frac{\partial\partial y}{\partial x^2},$$

per duplicem integrationem relationem inter x et y investigare.

Solutio.

Sit $e^{\lambda x}\partial x$ multiplicator hanc aequationem per se integrabilem reddens, eritque $A - B\lambda + C\lambda^2 = 0$, tum sumatur

$$A' = \frac{A}{\lambda} = B - C\lambda \text{ et } B' = \frac{B\lambda - A}{\lambda^2} = C,$$

positoque

$$e^{-\lambda x}\int e^{\lambda x} X \partial x = X',$$

aequatio semel integrata est

$$X' = A' y + B' \frac{\partial y}{\partial x}.$$

Hujus jam multiplicator sit $e^{\mu x}\partial x$, eritque $A' - B'\mu = 0$, ac statuatur $A'' = \frac{A'}{\mu} = B'$; positoque $e^{-\mu x}\int e^{\mu x} X' \partial x = X''$ habebimus $X'' = A'' y$, quae est aequatio bis integrata relationem quaesitam inter x et y exprimens.

Cum igitur hic sit $A'' = B'$ et $B' = C$, erit $A'' = C$. Deinde loco A' et B' substitutis valoribus, aequatio $A' - B'\mu = 0$ induit hanc formam

$$B - C\lambda - C\mu = 0, \text{ seu } B - C(\lambda + \mu) = 0,$$

ex qua cum sit $\lambda + \mu = \frac{B}{C}$, patet $\lambda + \mu$ aequari summae binarum radicum aequationis $A - B\lambda + C\lambda^2 = 0$. Quoniam igitur λ ejus una est radix, μ necessario ejus alteram radicem denotat. Quare si ex aequatione proposita, uti in capite praecedente fecimus, hanc formemus aequationem $A + Bz + Czz = 0$, ejus radices erunt $z = -\lambda$ et $z = -\mu$. Seu si factores ejus statuamus $C(\alpha + z)(\beta + z)$, litterae α et β praebebunt valores λ et μ. Hinc cum sit

$$X' = e^{-\alpha x}\int e^{\alpha x} X \partial x \text{ erit}$$

$$X'' = e^{-\beta x}\int e^{(\beta-\alpha)x}\partial x \int e^{\alpha x} X \partial x. \quad \text{At}$$

$$\int e^{(\beta-\alpha)x}\partial x\int e^{\alpha x}X\partial x = \frac{1}{\beta-\alpha}e^{(\beta-\alpha)x}\int e^{\alpha x}X\partial x - \frac{1}{\beta-\alpha}\int e^{\beta x}X\partial x;$$

unde concluditur

$$X'' = \frac{1}{\beta-\alpha}e^{-\alpha x}\int e^{\alpha x} X\partial x + \frac{1}{\alpha-\beta}e^{-\beta x}\int e^{\beta x}X\partial x.$$

Quocirca aequationis propositae integrale completum est

$$C y = \frac{1}{\beta-\alpha}e^{-\alpha x}\int e^{\alpha x} X\partial x + \frac{1}{\alpha-\beta}e^{-\beta x}\int e^{\beta x}X\partial x$$

ubi litterae α et β ita sunt capiendae, ut sit

$$A + Bz + Czz = C(\alpha + z)(\beta + z).$$

Corollarium 1.

1144. Si bini hi factores sint aequales, seu $\beta = \alpha$, erit

$$X'' = e^{-\alpha x}\int \partial x \int e^{\alpha x} X \partial x = e^{-\alpha x} x \int e^{\alpha x} X \partial x - e^{-\alpha x}\int e^{\alpha x} X x \partial x;$$

ideoque casu

$$A + Bz + Czz = C(\alpha + z)^2,$$

aequationis nostrae integrale est

$$Cy = e^{-\alpha x}(x\int e^{\alpha x} X \partial x - \int e^{\alpha x} X x \partial x).$$

Corollarium 2.

1145. Si bini factores sint imaginarii, quod evenit si

$A + Bz + Czz = C(ff + 2fz \cos.\theta + zz)$, erit

$\alpha = f(\cos.\theta + \sqrt{-1}.\sin.\theta)$ et

$\beta = f(\cos.\theta - \sqrt{-1}.\sin.\theta)$ hinc

$e^{\alpha x} = e^{fx\cos.\theta}(\cos.fx\sin.\theta + \sqrt{-1}.\sin.fx\sin.\theta)$ et

$e^{\beta x} = e^{fx\cos.\theta}(\cos.fx\sin.\theta - \sqrt{-1}.\sin.fx\sin.\theta)$, atque

$\beta - \alpha = -2\sqrt{-1}.f\sin.\theta.$

Corollarium 3.

1146. Quo haec facilius substituere queamus, sit brevitatis gratia

$$e^{fx\cos.\theta} = m, \quad \cos.fx\sin.\theta = p, \quad \text{et } \sin.fx\sin.\theta = q,$$

ut sit

$$e^{\alpha x} = mp + mq\sqrt{-1}, \quad \text{et } e^{\beta x} = mp - mq\sqrt{-1}.$$

Hinc fit

$$\int e^{\alpha x} X \partial x = \int mpX\partial x + \int mqX\partial x\sqrt{-1} \quad \text{et}$$
$$\int e^{\beta x} X \partial x = \int mpX\partial x - \int mqX\partial x\sqrt{-1}.$$

Tum vero est

$$e^{-\alpha x} = \frac{p - q\sqrt{-1}}{m}, \text{ et } e^{-\beta x} = \frac{p + q\sqrt{-1}}{m}.$$

Corollarium 4.

1147. Ex his colligimus

$$e^{-\alpha x}\int e^{\alpha x} X\,\partial x = \frac{p}{m}\int mpX\,\partial x - \frac{q\sqrt{-1}}{m}\int mpX\,\partial x$$
$$+ \frac{p\sqrt{-1}}{m}\int mqX\,\partial x + \frac{q}{m}\int mqX\,\partial x,$$

et sumto $\sqrt{-1}$ negativo, prodit $e^{-\beta x}\int e^{\beta x} X\,\partial x$, quae forma inde subtracta relinquit

$$-\frac{2q\sqrt{-1}}{m}\int mpX\,\partial x + \frac{2p\sqrt{-1}}{m}\int mqX\,\partial x;$$

hocque residuum dividi debet per

$$\beta - \alpha = -2\sqrt{-1}.f\sin.\theta.$$

Unde integrale colligitur

$$Cy = \frac{q}{m\sin.\theta}\int mpX\,\partial x - \frac{p}{m\sin.\theta}\int mqX\,\partial x.$$

Corollarium 5.

1148. Restituantur pro m, p, q valores assumti, atque aequationis nostrae, si fuerit

$$A + Bz + Czz = C(ff + 2fz\cos.\theta + zz)$$

integrale erit

$$Cy = e^{-fx\cos.\theta}\left(\frac{\sin.fx\sin.\theta}{\sin.\theta}\int e^{fx\cos.\theta}X\partial x\cos.fx\sin.\theta - \frac{\cos.fx\sin.\theta}{\sin.\theta}\int e^{fx\cos.\theta}X\partial x\sin.fx\sin.\theta\right),$$

quae ergo expressio aequivalet illi, si α et β valores imaginarios obtineant.

Problema 149.

1149. Proposita aequatione differentiali tertii gradus

$$X = Ay + B\frac{\partial y}{\partial x} + C\frac{\partial\partial y}{\partial x^2} + D\frac{\partial^3 y}{\partial x^3}$$

per triplicem integrationem ejus integrale completum invenire.

Solutio

Posito multiplicatore $e^{\lambda x}\partial x$, debet esse

$$A - B\lambda + C\lambda^2 - D\lambda^3 = 0,$$

tum sumatur

$$A' = B - C\lambda + D\lambda^2,\ B' = C - D\lambda \text{ et } C' = D,$$

positoque

$$e^{-\lambda x}\int e^{\lambda x} X\partial x = X',$$

aequatio semel integrata praebet

$$X' = A'y + B'\frac{\partial y}{\partial x} + C'\frac{\partial\partial y}{\partial x^2}.$$

Hujus porro multiplicator statuatur $e^{\mu x}\partial x$, ut sit

$$A' - B'\mu + C'\mu^2 = 0,$$

sumaturque

$$A'' = B' - C'\mu \text{ et } B'' = C',$$

et posito

$$e^{-\mu x}\int e^{\mu x} X'\partial x = X'',$$

aequatio secunda integralis est

$$X'' = A''y + B''\frac{\partial y}{\partial x},$$

cujus multiplicator erit $e^{\nu x}\partial x$, sumendo $A'' - B''\nu = 0$, at posito $A''' = B''$, erit aequatio integralis tertia

$$e^{-\nu x}\int e^{\nu x} X''\partial x = A'''y = Dy,$$

quaeri ergo oportet quantitates λ, μ, ν. Est vero primo

$$A - \beta\lambda + C\lambda^2 - D\lambda^3 = 0, \text{ tum}$$

$$B - C(\lambda + \mu) + D(\lambda\lambda + \lambda\mu + \mu\mu) = 0,$$

et ob

$$A'' = C - D(\lambda + \mu) \text{ et } B'' = D,$$

erit tertio

$$C - D(\lambda + \mu + \nu) = 0;$$

ex qua postrema aequalitate patet, $\lambda + \mu + \nu$ aequari summae radicum aequationis primae, cujus λ est una radix. Quod autem μ et ν sint reliquae radices, hoc modo ostenditur. Consideretur aequatio

$$A + Bz + Cz^2 + Dz^3 = 0,$$

cujus si una radix sit $z = -\lambda$, seu $\lambda + z$ unus factor, dividatur per eum aequatio, ac prodibit

$$Dz^2 + (C - D\lambda)z + B + C\lambda + D\lambda\lambda = 0,$$

quae est ipsa aequatio secunda $C'zz + B'z + A' = 0$, cujus radices sunt $z = -\mu$ et $z = -\nu$, seu factores $(\mu + z)(\nu + z)$, uti in problemate praecedente ostendimus. Quare si formulae

$$A + Bz + Cz^2 + Dz^3,$$

factores sint

$$D(\alpha + z)(\beta + z)(\gamma + z),$$

pro integrali ultimo inveniendo ponatur

$$e^{-\alpha x}\int e^{\alpha x} X \partial x = X', \quad e^{-\beta x}\int e^{\beta x} X' \partial x = X'', \text{ et}$$
$$e^{-\gamma x}\int e^{\gamma x} X'' \partial x = X''',$$

eritque $Dy = X'''$. Verum per reductionem integralium est, uti supra vidimus

$$X'' = \frac{1}{\beta - \alpha} e^{-\alpha x}\int e^{\alpha x} X \partial x + \frac{1}{\alpha - \beta} e^{-\beta x}\int e^{\beta x} X \partial x,$$

hincque porro

$$\int e^{\gamma x} X'' \partial x = \frac{1}{(\beta - \alpha)(\gamma - \alpha)} e^{(\gamma - \alpha)x}\int e^{\alpha x} X \partial x - \frac{1}{(\beta - \alpha)(\gamma - \alpha)}\int e^{\gamma x} X \partial x$$
$$+ \frac{1}{(\alpha - \beta)(\gamma - \beta)} e^{(\gamma - \beta)x}\int e^{\beta x} X \partial x - \frac{1}{(\alpha - \beta)(\gamma - \beta)}\int e^{\gamma x} X \partial x,$$

ubi bini postremi termini contrahuntur in

$$\frac{1}{(\alpha - \gamma)(\gamma - \beta)}\int e^{\gamma x} X \partial x.$$

Quamobrem integrale quaesitum est

$$Dy = \frac{e^{-\alpha x}\int e^{\alpha x} X \partial x}{(\beta-\alpha)(\gamma-\alpha)} + \frac{e^{-\beta x}\int e^{\beta x} X \partial x}{(\alpha-\beta)(\gamma-\beta)} + \frac{e^{-\gamma x}\int e^{\gamma x} X \partial x}{(\alpha-\gamma)(\beta-\gamma)}.$$

Corollarium 1.

1150. Si formulae $A + Bz + Cz^2 + Dz^3$ duo factores fuerint aequales, puta $\gamma = \beta$, erit

$$\int e^{\beta x} X'' \partial x = \frac{1}{(\beta-\alpha)^2} e^{(\beta-\alpha)x}\int e^{\alpha x} X \partial x - \frac{1}{(\beta-\alpha)^2}\int e^{\beta x} X \partial x$$
$$+ \frac{1}{\alpha-\beta} x \int e^{\beta x} X \partial x - \frac{1}{\alpha-\beta}\int e^{\beta x} X x \partial x,$$

ideoque integrale hoc casu erit

$$Dy = \frac{e^{-\alpha x}\int e^{\alpha x} X\partial x - e^{-\beta x}\int e^{\beta x} X \partial x}{(\beta-\alpha)^2} + \frac{e^{-\beta x} x \int e^{\beta x} X \partial x - e^{-\beta x}\int e^{\beta x} X x \partial x}{\alpha-\beta}.$$

Corollarium 2.

1151. Si omnes tres factores sint aequales, seu $\alpha = \beta = \gamma$, erit

$$e^{\alpha x} X'' = \int \partial x \int e^{\alpha x} X \partial x = x\int e^{\alpha x} X \partial x - \int e^{\alpha x} X x \partial x, \text{ et}$$
$$e^{\alpha x} X''' = \int e^{\alpha x} X'' \partial x = \int \partial x \int \partial x \int e^{\alpha x} X \partial x, \text{ seu}$$
$$e^{\alpha x} X''' = \tfrac{1}{2} x x \int e^{\alpha x} X \partial x - x \int e^{\alpha x} X x \partial x + \tfrac{1}{2}\int e^{\alpha x} X x x \partial x,$$

unde integrale hoc casu erit

$$Dy = \tfrac{1}{2} e^{-\alpha x}(x x \int e^{\alpha x} X \partial x - 2x \int e^{\alpha x} X x \partial x + \int e^{\alpha x} X x x \partial x),$$

seu

$$Dy = e^{-\alpha x}\int \partial x \int \partial x \int e^{\alpha x} X \partial x.$$

Scholion.

1152. In genere etiam nulla integralium reductione adhibita integrale nostrae aequationis ita exprimi potest, ut sit

$$Dy = e^{-\gamma x}\int e^{(\gamma-\beta)x}\partial x \int e^{(\beta-\alpha)x}\partial x \int e^{\alpha x} X \partial x,$$

posito

$$A + Bz + Cz^2 + Dz^3 = D(\alpha+z)(\beta+z)\gamma+z),$$

ubi imprimis notatu dignum occurrit, quod ternas litteras α, β, γ quomodocunque inter se permutare licet, ita ut haec integralis expressio sex modis variari possit. In problemate etiam praecedente, ubi duo tantum factores occurrunt

$$C(\alpha+z)(\beta+z) = A + Bz + Cz^2,$$

aequationis

$$X = Ay + B\frac{\partial y}{\partial x} + C\frac{\partial\partial y}{\partial x^2}$$

integrale completum ita exhiberi potest

$$Cy = e^{-\beta x}\int e^{(\beta-\alpha)x}\partial x\int e^{\alpha x}X\partial x,$$

ac permutatis litteris α et β etiam hoc modo

$$Cy = e^{-\alpha x}\int e^{(\alpha-\beta)x}\partial x\int e^{\beta x}X\partial x.$$

Quarum formularum aequalitas si fuerit perspecta, id quod tentanti facile patebit, praecedentium quoque variationem declarat. Sit enim $e^{-\alpha x}\int e^{\alpha x}X\partial x = X'$, erit pro superiori formula

$$Dy = e^{-\gamma x}\int e^{(\gamma-\beta)x}\partial x\int e^{\beta x}X'\partial x,$$

cui cum aequalis sit ista

$$Dy = e^{-\beta x}\int e^{(\beta-\gamma)x}\partial x\int e^{\gamma x}X'\partial x,$$

erit etiam pro X' valore restituto

$$Dy = e^{-\beta x}\int e^{(\beta-\gamma)x}\partial x\int e^{(\gamma-\alpha)x}\partial x\int e^{\alpha x}X\partial x,$$

quae a prima hoc tantum differt, quod litterae β et γ sunt permutatae. Quod autem etiam litterae β et γ cum α permutari queant, hoc modo difficilius ostenditur, ex reductione autem in solutione adhibita, atque adeo ex ipsa solutionis indole per se est manifestum.

Problema 150.

1153. Proposita aequatione differentiali quarti gradus

$$X = Ay + B\frac{\partial y}{\partial x} + C\frac{\partial\partial y}{\partial x^2} + D\frac{\partial^3 y}{\partial x^3} + E\frac{\partial^4 y}{\partial x^4},$$

sumto elemento ∂x constante, et denotante X functionem quamcunque ipsius x, ejus integrale investigare.

Solutio.

In subsidium vocetur formula algebraica ex aequatione proposita facile formanda

$$A + Bz + Czz + Dz^3 + Ez^4 = P,$$

quae in factores suos simplices resolvatur, ut sit

$$P = E(\alpha + z)(\beta + z)(\gamma + z)(\delta + z),$$

et multiplicator aequationem nostram integrabilem reddens erit $e^{\lambda x}\partial x$, sumendo λ aequali uni litterarum α, β, γ, δ; sumatur ergo $\lambda = \alpha$, ut sit multiplicator $e^{\alpha x}\partial x$, atque posito $e^{-\alpha x}\int e^{\alpha x} X \partial x = X'$, aequatio semel integrata erit

$$X' = A'y + B'\frac{\partial y}{\partial x} + C'\frac{\partial \partial y}{\partial x^2} + D'\frac{\partial^3 y}{\partial x^3},$$

ubi A', B', C', D' ita determinantur, ut sit

$$A' = \frac{A}{\alpha},\ B' = \frac{-A + B\alpha}{\alpha^2},\ C' = \frac{C\alpha^2 - B\alpha + A}{\alpha^3},\ D' = \frac{D\alpha^3 - C\alpha^2 + B\alpha - A}{\alpha^4},$$

seu

$$A' = \frac{A}{\alpha},\ B' = \frac{B - A'}{\alpha},\ C' = \frac{C - B'}{\alpha},\ D' = \frac{D - C'}{\alpha},$$

vel etiam

$$A = A'\alpha,\ B = B'\alpha + A',\ C = C'\alpha + B',\ D = D'\alpha + C'.$$

Ex quibus determinationibus liquet, si ponatur

$$A' + B'z + C'z^2 + D'z^3 = Q,$$

hanc formulam Q nasci ex formula P, si haec per $\alpha + z$ dividatur, ita ut sit

$$Q = \frac{P}{\alpha + z} = E(\beta + z)(\gamma + z)(\delta + z).$$

Eodem ergo modo secundam integrationem instituamus ope multiplicatoris $e^{\beta x}\partial x$, et posito

$$e^{-\beta x}\int e^{\beta x} X' \partial x = X'',$$

erit aequatio integralis

$$X'' = A'' y + B'' \frac{\partial y}{\partial x} + C'' \frac{\partial \partial y}{\partial x^2},$$

coëfficientibus A'', B'', C'' ita sumtis, ut sit

$$A'' + B'' z + C'' z^2 = \frac{P}{(\alpha + z)(\beta + z)} = E(\gamma + z)(\delta + z).$$

Hinc porro ope multiplicatoris $e^{\gamma x} \partial x$ integrando, si ponamus $e^{-\gamma x} \int e^{\gamma x} X'' \partial x = X'''$, inveniemus

$$X''' = A''' y + B''' \frac{\partial y}{\partial x},$$

existente

$$A''' + B''' z = \frac{P}{(\alpha + z)(\beta + z)(\gamma + z)} = E(\delta + z).$$

Ac tandem ope multiplicatoris $e^{\delta x} \partial x$, posita forma

$$e^{-\delta x} \int e^{\delta x} X''' \partial x = X'''',$$

integrale ultimum reperitur

$$X'''' = A'''' y \text{ existente } A'''' = E.$$

Haec igitur omnia colligendo, integrale quaesitum erit

$$E y = e^{-\delta x} \int e^{(\delta - \gamma) x} \partial x \int e^{(\gamma - \beta) x} \partial x \int e^{(\beta - \alpha) x} \partial x \int e^{\alpha x} X \partial x,$$

quae expressio jam sine ullis ambagibus ex resolutione formae principalis

$$P = A + B z + C z^2 + D z^3 + E z^4,$$

in factores scilicet

$$P = E(\alpha + z)(\beta + z)(\gamma + z)(\delta + z),$$

confici potest, ubi notandum quomodocunque ordo litterarum $\alpha, \beta, \gamma, \delta$ permutetur, pro $E y$ semper eundem valorem prodire debere.

Corollarium 1.

1154. Cum sit $X' = e^{-\alpha x} \int e^{\alpha x} X \partial x$, erit uti jam vidimus:

$$X'' = e^{-\beta x}\int e^{\beta x} X' \partial x = e^{-\beta x}\left(\frac{e^{(\beta-\alpha)x}}{\beta-\alpha}\int e^{\alpha x} X \partial x - \frac{1}{\beta-\alpha}\int e^{\beta x} X \partial x\right),$$

seu

$$X'' = \frac{e^{-\alpha x}\int e^{\alpha x} X \partial x}{\beta-\alpha} + \frac{e^{-\beta x}\int e^{\beta x} X \partial x}{\alpha-\beta}.$$

Corollarium 2.

1155. Porro ob $X''' = e^{-\gamma x}\int e^{\gamma x} X'' \partial x$, erit simili modo reductionem instituendo

$$X''' = \frac{e^{-\alpha x}\int e^{\alpha x} X \partial x}{(\beta-\alpha)(\gamma-\alpha)} + \frac{e^{-\gamma x}\int e^{\gamma x} X \partial x}{(\beta-\alpha)(\alpha-\gamma)} + \frac{e^{-\beta x}\int e^{\beta x} X \partial x}{(\alpha-\beta)(\gamma-\beta)} + \frac{e^{-\gamma x}\int e^{\gamma x} X \partial x}{(\alpha-\beta)(\beta-\gamma)},$$

quae reducitur ad hanc formam

$$X''' = \frac{e^{-\alpha x}\int e^{\alpha x} X \partial x}{(\beta-\alpha)(\gamma-\alpha)} + \frac{e^{-\beta x}\int e^{\beta x} X \partial x}{(\alpha-\beta)(\gamma-\beta)} + \frac{e^{-\gamma x}\int e^{\gamma x} X \partial x}{(\alpha-\gamma)(\beta-\gamma)}.$$

Corollarium 3.

1156. Hinc simili modo evolvitur valor X'''', ubi quidem sufficeret primum membrum eruisse, quippe ex quo ob permutabilitatem reliqua sponte formantur. Hoc modo integrale nostrae aequationis reperietur hac forma expressum

$$Ey = \frac{e^{-\alpha x}\int e^{\alpha x} X \partial x}{(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)} + \frac{e^{-\beta x}\int e^{\beta x} X \partial x}{(\alpha-\beta)(\gamma-\beta)(\delta-\beta)} + \frac{e^{-\gamma x}\int e^{\gamma x} X \partial x}{(\alpha-\gamma)(\beta-\gamma)(\delta-\gamma)} + \frac{e^{-\delta x}\int e^{\delta x} X \partial x}{(\alpha-\delta)(\beta-\delta)(\gamma-\delta)}.$$

Scholion.

1157. Si duae pluresve radices sint aequales vel imaginariae, integralia inventa transformationem postulant, quam deinceps

investigabimus. Atque haec postrema quidem forma magis apta videtur, unde transformationes repetantur. Ita pro factorum aequalitate si sit $\delta = \gamma$, bina postrema membra tantum reductionem postulant, ad quam inveniendam ponatur $\delta = \gamma - \omega$, et penultimum membrum erit $-\frac{e^{-\gamma x}\int e^{\gamma x} X \partial x}{\omega(\alpha-\gamma)(\beta-\gamma)}$; pro ultimo autem notandum est, esse

$$\frac{1}{\alpha-\delta} = \frac{1}{\alpha-\gamma+\omega} = \frac{1}{\alpha-\gamma} - \frac{\omega}{(\alpha-\gamma)^2}, \text{ et } \frac{1}{\beta-\delta} = \frac{1}{\beta-\gamma} - \frac{\omega}{(\beta-\gamma)^2},$$

hincque

$$\frac{1}{(\alpha-\delta)(\beta-\delta)} = \frac{1}{(\alpha-\gamma)(\beta-\gamma)} + \frac{\omega(2\gamma-\alpha-\beta)}{(\alpha-\gamma)^2(\beta-\gamma)^2}, \text{ unde}$$

$$\frac{1}{(\alpha-\delta)(\beta-\delta)(\gamma-\delta)} = \frac{1}{\omega(\alpha-\gamma)(\beta-\gamma)} + \frac{2\gamma-\alpha-\beta}{(\alpha-\gamma)^2(\beta-\gamma)^2}.$$

Tum vero pro numeratore erit

$$e^{-\delta x} = e^{-\gamma x}(1+\omega x), \text{ et } e^{\delta x} = e^{\gamma x}(1-\omega x),$$

ideoque

$$e^{-\delta x}\int e^{\delta x} X \partial x = e^{-\gamma x}\int e^{\gamma x} X \partial x + \omega e^{-\gamma x} x \int e^{\gamma x} X \partial x - \omega e^{-\gamma x}\int e^{\gamma x} X x \partial x,$$

atque hinc bina ultima membra ob terminos per ω divisos se destruentes, abeunt in hanc formam

$$\frac{(2\gamma-\alpha-\beta)e^{-\gamma x}\int e^{\gamma x} X \partial x}{(\alpha-\gamma)^2(\beta-\gamma)^2} + \frac{e^{-\gamma x} x \int e^{\gamma x} X \partial x - e^{-\gamma x}\int e^{\gamma x} X x \partial x}{(\alpha-\gamma)(\beta-\gamma)},$$

quae expressio etiam ex priori forma elicitur. Eodem modo problema in genere resolvi potest.

Problema 151.

1158. Proposita aequatione differentiali cujuscunque gradus

$$X = Ay + B\frac{\partial y}{\partial x} + C\frac{\partial \partial y}{\partial x^2} + D\frac{\partial^3 y}{\partial x^3} + \ldots + N\frac{\partial^n y}{\partial x^n},$$

sumto elemento ∂x constante, et denotante X functionem quamcunque ipsius x, ejus integrale investigare.

Solutio.

Formetur ex hac aequatione formula algebraica

$$A + B z + C z^2 + D z^3 + \ldots\ldots + N z^n = P,$$

quae in factores simplices resolvatur, ut sit

$$P = N(\alpha + z)(\beta + z)(\gamma + z)\ldots\ldots(\nu + z),$$

quorum numerus est n. Quodsi jam simili modo per singulas integrationes continuo progrediamur, tandem ad hanc aequationem integralem extremam perveniemus

$$N y = e^{-\nu x}\int e^{(\nu-\mu)x}\partial x\int e^{(\mu-\lambda)x}\partial x\int\ldots\ldots\int e^{(\beta-\alpha)x}\partial x\int e^{\alpha x}X\partial x,$$

seu cum factores inter se permutare liceat, erit etiam

$$N y = e^{-\alpha x}\int e^{(\alpha-\beta)x}\partial x\int e^{(\beta-\gamma)x}\partial x\int\ldots\ldots\int e^{(\mu-\nu)x}\partial x\int e^{\nu x}X\partial x.$$

Haec vero expressio per similes reductiones, quibus supra sumus usi, in sequentes partes resolvi potest, ad quas commodius repraesentandas sit brevitatis gratia

$$(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)\ldots\ldots(\nu-\alpha) = \alpha',$$
$$(\alpha-\beta)(\gamma-\beta)(\delta-\beta)\ldots\ldots(\nu-\beta) = \beta',$$
$$(\alpha-\gamma)(\beta-\gamma)(\delta-\gamma)\ldots\ldots(\nu-\gamma) = \gamma'$$

$$\vdots$$

$(\alpha-\nu)(\beta-\nu)(\gamma-\nu)\ldots\ldots(\mu-\nu) = \nu'$, hincque erit

$$N y = \frac{1}{\alpha'}e^{-\alpha x}\int e^{\alpha x}X\partial x + \frac{1}{\beta'}e^{-\beta x}\int e^{\beta x}X\partial x$$
$$+ \frac{1}{\gamma'}e^{-\gamma x}\int e^{\gamma x}X\partial x + \ldots\ldots + \frac{1}{\nu'}e^{-\nu x}\int e^{\nu x}X\partial x.$$

Ne autem opus sit ad valores α', β', γ', etc. inveniendos, tot factores in se invicem multiplicare, cum sit

$$\frac{D}{N(\alpha+z)} = (\beta+z)(\gamma+z)(\delta+z)\ldots\ldots(\nu+z),$$

evidens est, hanc formulam praebere valorem α', si in ea statuatur $z = -\alpha$; hoc autem casu fractionis $\frac{P}{N(\alpha+z)}$ tam numerator, quam denominator evanescit, ex quo ejus valor erit $\frac{\partial P}{N \partial z}$. Quare cum sit

$$P = A + Bz + Cz^2 + Dz^3 + \ldots\ldots + Nz^n, \text{ erit}$$
$$\frac{\partial P}{\partial z} = B + 2Cz + 3Dz^2 + \ldots\ldots + nNz^{n-1},$$

quae expressio vocetur Q, unde patet fore

$$\alpha' = \frac{Q}{N} \text{ posito } z = -\alpha,$$
$$\beta' = \frac{Q}{N} \text{ posito } z = -\beta,$$
$$\gamma' = \frac{Q}{N} \text{ posito } z = -\gamma,$$
etc.

Vel cum his valoribus substitutis aequatio integralis per N dividi queat, sequentes valores colligantur

$$B - 2C\alpha + 3D\alpha^2 - 4E\alpha^3 \ldots\ldots \pm nN\alpha^{n-1} = \mathfrak{A},$$
$$B - 2C\beta + 3D\beta^2 - 4E\beta^3 \ldots\ldots \pm nN\beta^{n-1} = \mathfrak{B},$$
$$B - 2C\gamma + 3D\gamma^2 - 4E\gamma^3 \ldots\ldots \pm nN\gamma^{n-1} = \mathfrak{C},$$
$$\vdots$$
$$B - 2C\nu + 3D\nu^2 - 4E\nu^3 \ldots\ldots \pm nN\nu^{n-1} = \mathfrak{N},$$

quibus constitutis erit integrale quaesitum

$$y = \frac{1}{\mathfrak{A}} e^{-\alpha x} \int e^{\alpha x} X \partial x + \frac{1}{\mathfrak{B}} e^{-\beta x} \int e^{\beta x} X \partial x + \frac{1}{\mathfrak{C}} e^{-\gamma x} \int e^{\gamma x} X \partial x + \text{etc.}$$

quoad omnes factores fuerint in computum ducti.

Corollarium 1.

1159. Cum sit

$$\alpha' = \frac{\mathfrak{A}}{N}, \; \beta' = \frac{\mathfrak{B}}{N}, \; \gamma' = \frac{\mathfrak{C}}{N}, \text{ etc. erit}$$
$$\mathfrak{A} = N\alpha', \; \mathfrak{B} = N\beta', \; \mathfrak{C} = N\gamma', \text{ etc.}$$

Hinc ob

$$P = A + Bz + Cz^2 + Dz^3 + \ldots\ldots + Nz^n = N(\alpha + z)(\beta + z)(\gamma + z)\ldots\ldots(\nu + z),$$

erit

$$\mathfrak{A} = \frac{P}{\alpha + z} \text{ posito } z = -\alpha,$$

$$\mathfrak{B} = \frac{P}{\beta + z} \text{ posito } z = -\beta,$$

$$\mathfrak{C} = \frac{P}{\gamma + z} \text{ posito } z = -\gamma,$$

et ita porro.

Corollarium 2.

1160. Regula ergo hujus aequationis propositae

$$X = Ay + B\frac{\partial y}{\partial x} + C\frac{\partial \partial y}{\partial x^2} + D\frac{\partial^3 y}{\partial x^3} + \ldots\ldots + N\frac{\partial^n y}{\partial x^n},$$

integrale completum inveniendi ita se habet. Formetur inde formula algebraica haec

$$A + Bz + Cz^2 + Dz^3 + \ldots\ldots + Nz^n = P,$$

cujus quaerantur omnes factores simplices, qui sint

$$\alpha + z,\ \beta + z,\ \gamma + z,\ \delta + z, \text{ etc.}$$

quorum multitudo numero n est aequalis, tum pro singulis his factoribus sequentes quantitates constantes definiantur $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, etc. ut sit

$$\mathfrak{A} = \frac{P}{\alpha + z} \text{ posito } z = -\alpha, \text{ seu}$$

$$\mathfrak{A} = B - 2C\alpha + 3D\alpha^2 - 4E\alpha^3 \ldots\ldots \pm nN\alpha^{n-1},$$

$$\mathfrak{B} = \frac{P}{\beta + z} \text{ posito } z = -\beta, \text{ seu}$$

$$\mathfrak{B} = B - 2C\beta + 3D\beta^2 - 4E\beta^3 \ldots\ldots \pm nN\beta^{n-1},$$

$$\mathfrak{C} = \frac{P}{z + \gamma} \text{ posito } z = -\gamma, \text{ seu}$$

$$\mathfrak{C} = B - 2C\gamma + 3D\gamma^2 - 4E\gamma^3 \ldots\ldots \pm nN\gamma^{n-1},$$

his omnibus inventis erit integrale quaesitum

$$y = \frac{1}{\mathfrak{A}} e^{-\alpha x} \int e^{\alpha x} X \partial x + \frac{1}{\mathfrak{B}} e^{-\beta x} \int e^{\beta x} X \partial x + \frac{1}{\mathfrak{C}} e^{-\gamma x} \int e^{\gamma x} X \partial x + \text{etc.}$$

quae forma tot constat partibus, quot fuerint factores simplices in formula P.

Corollarium 3.

1161. Cum hoc modo integrale tot constet partibus, quoti ordinis est aequatio differentialis proposita, et una quaeque pars per integrationem unam invehat constantem arbitrariam; manifestum est integrale ope hujus regulae inventum fore completum.

Scholion.

1162. Integratio ergo hujusmodi aequationum differentialium nulla amplius laborat difficultate, si modo formulae illius algebraicae P omnes factores simplices, seu quod eodem redit, hujus aequationis algebraicae

$$A + Bz + Cz^2 + Dz^3 + \ldots\ldots\ldots + Nz^n = 0,$$

omnes radices numero n assignari queant. Hic vero duplicis generis casus occurrunt, quibus haec integratio vehementer impeditur, quando scilicet vel duo pluresve eorum factorum simplicium inter se fiunt aequales, vel imaginarii, quo quidem posteriori casu hoc tantum incommodi accedit, quod partes quaepiam integralis inventi imaginaria involvant, quae autem facta reductione se mutuo destruunt. Priori vero casu partes ex factoribus aequalibus oriundae adeo fiunt infinitae, sed ita diversis signis affectae, ut conjunctim nihilominus quantitatem finitam referant, cujus valorem nonnisi per plures ambages elicere licet, ubi probe notandum est, utroque casu inventionem reliquarum integralis partium, quae factoribus inaequalibus conveniunt, neutiquam hinc turbari. Methodum autem huic fini accommodatam in sequenti problemate explicabo.

Problema 162.

1163. Proposita aequatione differentiali cujuscunque gradus

$$X = A y + B \frac{\partial y}{\partial x} + C \frac{\partial \partial y}{\partial x^2} + D \frac{\partial^3 y}{\partial x^3} + E \frac{\partial^4 y}{\partial x^4} + \ldots + N \frac{\partial^n y}{\partial x^n},$$

si forma algebraica inde facta

$$P = A + B z + C z^2 + D z^3 + E z^4 + \ldots + N z^n$$

duos pluresve factores simplices inter se habeat aequales, partem integralis inde oriundam investigare.

Solutio.

Sint primo duo factores $\alpha + z$ et $\beta + z$ inter se aequales, seu $\beta = \alpha$, reliquus vero factor formae P sit $= Q$, ut habeatur

$$P = (\alpha + z)(\beta + z) Q = (\alpha + z)^2 Q,$$

posito autem $z = -\alpha$, abeat Q in $\mathfrak{C}$. Jam initio saltem litterae α et β ut diversae spectentur, excepta quantitate $\mathfrak{C}$ quae utrinque sit eadem, atque pro binis integralis partibus ex his binis factoribus oriundis habebimus

$$\mathfrak{A} = (\beta - \alpha)\mathfrak{C} \text{ et } \mathfrak{B} = (\alpha - \beta)\mathfrak{C}.$$

Partes autem integralis inde oriundae littera v designentur, ut sit

$$(\beta - \alpha)\mathfrak{C} v = e^{-\alpha x}\int e^{\alpha x} X \partial x - e^{-\beta x}\int e^{\beta x} X \partial x;$$

unde differentiando colligimus

$$(\beta - \alpha)\mathfrak{C}\partial v = -\alpha e^{-\alpha x}\partial x \int e^{\alpha x} X \partial x + \beta e^{-\beta x}\partial x \int e^{\beta x} X \partial x,$$

ad hanc addatur prior per $\beta \partial x$ multiplicata, fietque

$$(\beta - \alpha)\mathfrak{C}\partial v + (\beta - \alpha)\mathfrak{C}\beta v \partial x = (\beta - \alpha) e^{-\alpha x}\partial x \int e^{\alpha x} X \partial x,$$

quae per $\beta - \alpha$ divisa, et per $e^{\alpha x}$ multiplicata, ob $\beta = \alpha$, integrale praebet

$$\mathfrak{C} e^{\alpha x} v = \int \partial x \int e^{\alpha x} X \partial x.$$

Quocirca loco binarum partium ex factoribus aequalibus $\alpha + z$ et $\beta + z$ oriundarum scribi oportet hanc formulam

$$v = \tfrac{1}{\mathfrak{C}} e^{-\alpha x} \int \partial x \int e^{\alpha x} X \partial x,$$

ubi $\mathfrak{C}$ oritur ex forma $\frac{P}{(\alpha + z)^2}$ posito $z = -\alpha$.

Ponamus jam formulam P tres habere factores simplices aequales, ut sit $\alpha + z = \beta + z = \gamma + z$, quosquidem initio ut diversos spectemus.

Ponamus ergo $P = (\alpha + z)(\beta + z)(\gamma + z) Q$, abeatque Q $\mathfrak{M}$, posito $z = -\alpha$, ac pro integralis partibus habebimus

$$\mathfrak{A} = (\beta - \alpha)(\gamma - \alpha)\mathfrak{M}, \quad \mathfrak{B} = (\alpha - \beta)(\gamma - \beta)\mathfrak{M},$$
$$\mathfrak{C} = (\alpha - \gamma)(\beta - \gamma)\mathfrak{M}.$$

Hinc si summam trium integralis partium, quam quaerimus, littera v denotemus, erit

$$\mathfrak{M} v = \frac{e^{-\alpha x} \int e^{\alpha x} X \partial x}{(\beta - \alpha)(\gamma - \alpha)} + \frac{e^{-\beta x} \int e^{\beta x} X \partial x}{(\alpha - \beta)(\gamma - \beta)} + \frac{e^{-\gamma x} \int e^{\gamma x} X \partial x}{(\alpha - \gamma)(\beta - \gamma)}.$$

Cum nunc sit

$$\frac{1}{(\beta - \alpha)(\gamma - \alpha)} + \frac{1}{(\alpha - \beta)(\gamma - \beta)} + \frac{1}{(\alpha - \gamma)(\beta - \gamma)} = 0,$$

erit differentiando

$$\frac{\mathfrak{M} \partial v}{\partial x} = \frac{-\alpha e^{-\alpha x} \int e^{\alpha x} X \partial x}{(\beta - \alpha)(\gamma - \alpha)} - \frac{\beta e^{-\beta x} \int e^{\beta x} X \partial x}{(\alpha - \beta)(\gamma - \beta)} - \frac{\gamma e^{-\gamma x} \int e^{\gamma x} X \partial x}{(\alpha - \gamma)(\beta - \gamma)},$$

ad quam si prima per α multiplicata addatur, fit

$$\mathfrak{M}\left(\frac{\partial v}{\partial x} + \alpha v\right) = \frac{e^{-\beta x} \int e^{\beta x} X \partial x}{\gamma - \beta} + \frac{e^{-\gamma x} \int e^{\gamma x} X \partial x}{\beta - \gamma}.$$

Haec aequatio denuo differentietur, ut prodeat

$$\mathfrak{M}\left(\frac{\partial \partial v}{\partial x^2} + \frac{\alpha \partial v}{\partial x}\right) = \frac{-\beta e^{-\beta x} \int e^{\beta x} X \partial x}{\gamma - \beta} - \frac{\gamma e^{-\gamma x} \int e^{\gamma x} X \partial x}{\beta - \gamma},$$

ad quam illa per $\beta = \alpha$ multiplicata, si addatur, oritur

$$\mathfrak{M}\left(\frac{\partial\partial v}{\partial x^2} + \frac{2\alpha\partial v}{\partial x} + \alpha\alpha v\right) = e^{-\gamma x}\int e^{\gamma x} X \partial x = e^{-\alpha x}\int e^{\alpha x} X \partial x;$$

unde jam omnia incommoda sunt sublata. Multiplicetur nunc per $e^{\alpha x}\partial x$, et integratio dabit

$$\mathfrak{M} e^{\alpha x}\left(\frac{\partial v}{\partial x} + \alpha v\right) = \int \partial x \int e^{\alpha x} X \partial x,$$

quae per ∂x multiplicata denuo fit integrabilis, proditque

$$\mathfrak{M} e^{\alpha x} v = \int \partial x \int \partial x \int e^{\alpha x} X \partial x.$$

Quocirca si forma P factorem habeat cubicum $(\alpha + z)^3$, quaeratur quantitas $\mathfrak{M}$, ut sit

$$\mathfrak{M} = \frac{P}{(\alpha + z)^3}, \text{ posito } z = -\alpha,$$

et integralis pars hinc oriunda erit

$$\frac{1}{\mathfrak{M}} e^{-\alpha x}\int \partial x \int \partial x \int e^{\alpha x} X \partial x.$$

Simili modo si formula P quatuor habeat factores aequales, ut sit $P = (\alpha + z)^4 Q$, capiatur $\mathfrak{M} = \frac{P}{(\alpha + z)^4}$ seu $\mathfrak{M} = Q$, posito $z = -\alpha$, et integralis pars inde nata erit

$$\frac{1}{\mathfrak{M}} e^{-\alpha x}\int \partial x \int \partial x \int \partial x \int e^{\alpha x} X \partial x:$$

sicque etiam casus, quibus formula P adhuc plures habet factores aequales, facile resolventur.

Nota. Tota haec solutio est vitiosa, propterea quod licet quantitates α, β, γ, etc. quae ponuntur aequales, ut diversae spectentur, tamen pro singulis membris quantitas $\mathfrak{M}$ eundem valorem retinere assumitur. Quodsi enim litterae α, β, γ, etc. infinite parum a se invicem discrepare concipiantur, etiam in valoribus litterа $\mathfrak{M}$ indicatis differentiam infinite parvam agnoscere oportet, unde cum singulae partes integralis fiant infinitae, iisque evolutis membra infinita se mutuo tollant, ex differentiis infinite parvis litterae $\mathfrak{M}$ partes quoque finitae emergunt. Correctionem horum errorum petere licet ex seq. Probl. 154, dum factores aequales in aequationem peculiarem conjiciuntur. Malui autem hunc correctionis laborem industriae lectorum relinquere, quam hoc opus a tali errore liberare, saepe enim plus prodest errores, in quos etiam exercitatis incidere contingit, conservari, quo melius harum rerum studiosi addiscant, quanta circumspectione cavendum sit, ne in ratiocinando hallucinemur.

Corollarium 1.

1164. Notatu hic omnino est dignum, quod hae formulae

$$\partial v + \alpha v \partial x, \quad \partial\partial v + 2\alpha \partial x \partial v + \alpha^2 v \partial x^2,$$
$$\partial^3 v + 3\alpha \partial x \partial\partial v + 3\alpha^2 \partial x^2 \partial v + \alpha^3 v^3 \partial x^3$$

et in genere haec

$$\partial^n v + \frac{n}{1}\alpha \partial x \partial^{n-1} v + \frac{n(n-1)}{1.2}\alpha^2 \partial x^2 \partial^{n-2} v + \frac{n(n-1)(n-2)}{1.2.3}\alpha^3 \partial x^3 \partial^{n-3} v + \text{etc.}$$

si semel per $e^{\alpha x}$ multiplicentur, successive toties integrationem admittant, quot unitates continet index n, ita ut postremum integrale sit $e^{\alpha x}$.

Corollarium 2.

1165. Ratio autem hujus phaenomeni inde est manifesta, quod si formula $e^{\alpha x} v$ continuo differentietur, sumto elemento ∂x constante, formulae illae differentiales per $e^{\alpha x}$ multiplicatae prodeant, ita ut sit

$$\partial^n . e^{\alpha x} v = e^{\alpha x}\left(\partial^n v + \frac{n}{1}\alpha \partial x \partial^{n-1} v + \frac{n(n-1)}{1.2}\alpha^2 \partial x^2 \partial^{n-2} v + \text{etc.}\right)$$

Corollarium 3.

1166. Aeque memoratu dignum est alterum phaenomenum, quod solutio ista nobis offert; sumtis scilicet numeris quibuscunque α, β, γ, δ, etc. sequentes aequalitates semper locum habere, ut sit

$$\frac{1}{\alpha-\beta} + \frac{1}{\beta-\alpha} = 0,$$
$$\frac{1}{(\alpha-\beta)(\alpha-\gamma)} + \frac{1}{(\beta-\alpha)(\beta-\gamma)} + \frac{1}{(\gamma-\alpha)(\gamma-\beta)} = 0,$$
$$\frac{1}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{1}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{1}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{1}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)} = 0,$$
etc.

quotcunque numeri hoc modo capiantur.

Corollarium 4.

1167. Si formula P in factores simplices resoluta ponatur

$$P = N(\alpha + z)(\beta + z)(\gamma + z) \ldots\ldots\ldots (\mu + z)(\nu + z);$$

expressio integralis prius inventa (1158.) quae erat

$$Ny = e^{-\alpha x}\int e^{(\alpha-\beta)x}\partial x\int e^{(\beta-\gamma)x}\partial x\int e^{(\gamma-\delta)x}\partial x \ldots\ldots \int e^{(\mu-\nu)x}\partial x\int e^{\nu x}X\partial x,$$

ob factores aequales nulla implicatur difficultate, forma autem posterior, qua integrale in partes ex singulis factoribus ortas distributum exhibetur, et quae ad usum multo magis accommodata videtur, eo difficiliori egebat evolutione.

Scholion.

1168. Phaenomenum Corollario 3. observatum eo majorem attentionem meretur, quod etiam ad Arithmeticam vulgarem transferri potest, ubi usu adeo insigni non cariturum videtur, praecipue quod ejus demonstratio minime sit obvia, sed ex profundioribus Analyseos penetralibus repeti debeat, ex quo haud alienum fore arbitror, si huic insigni Theoremati arithmetico hic locum concedam, idque eo magis, quod solutio problematis hic exposita sine demonstratione istius Theorematis minime foret perfecta.

Theorema Arithmeticum.

1169. Si habeantur numeri quotcunque a, b, c, d, etc. ex iisque dum a quolibet singuli reliqui subtrahantur, formentur sequentia producta

$$(a-b)(a-c)(a-d)(a-e) \text{ etc.} = \mathfrak{A}$$
$$(b-a)(b-c)(b-d)(b-e) \text{ etc.} = \mathfrak{B}$$
$$(c-a)(c-b)(c-d)(c-e) \text{ etc.} = \mathfrak{C}$$
$$(d-a)(d-b)(d-c)(d-e) \text{ etc.} = \mathfrak{D}$$

etc.

semper habebitur

$$\frac{1}{\mathfrak{A}} + \frac{1}{\mathfrak{B}} + \frac{1}{\mathfrak{C}} + \frac{1}{\mathfrak{D}} + \text{etc.} = 0.$$

Demonstratio.

Consideretur secundum principia in Introductione ad Analysin infinitorum tradita haec fractio

$$\frac{Z}{(z-a)(z-b)(z-c)(z-d)\ \text{etc.}},$$

ubi Z denotet ejusmodi functionem rationalem integram ipsius z, in qua summa potestas ipsius z minor sit numero factorum denominatoris; haecque fractio resolvi poterit in has fractiones simplices, quibus ea junctim sumtis sit aequalis, scililet

$$\frac{A}{z-a} + \frac{B}{z-b} + \frac{C}{z-c} + \frac{D}{z-d} + \text{etc.}$$

Ad quam resolutionem sumamus illum numeratorem $Z = z^n$, existente n numero integro minore quam denominator continet factores, atque hi numeratores ita definiuntur, ut sit

$$A = \frac{a^n}{(a-b)(a-c)(a-d)\ \text{etc.}}$$

$$B = \frac{b^n}{(b-a)(b-c)(b-d)\ \text{etc.}}$$

$$C = \frac{c^n}{(c-a)(c-b)(c-d)\ \text{etc.}}$$

etc.

Cum igitur istae fractiones negative sumtae nempe

$$\frac{A}{a-z} + \frac{B}{b-z} + \frac{C}{c-z} + \frac{D}{d-z} + \text{etc.}$$

ad fractionem propositam adjectae in nihilum abeant, si z sit numerorum propositorum a, b, c, d, etc. ultimus, quorum adeo multitudo major est quam $n+1$, ponatur

$$(a-b)(a-c)(a-d)\ldots\ldots(a-z)=\mathfrak{A}$$
$$(b-a)(b-c)(b-d)\ldots\ldots(b-z)=\mathfrak{B}$$
$$(c-a)(c-b)(c-d)\ldots\ldots(c-z)=\mathfrak{C}$$
$$(d-a)(d-b)(d-c)\ldots\ldots(d-z)=\mathfrak{D}$$
etc.
$$(z-a)(z-b)(z-c)\ldots\ldots(z-y)=\mathfrak{Z}$$

ut fractio proposita sit $\frac{z^n}{\mathfrak{Z}}$. Atque hinc perspicuum est, summam omnium harum fractionum esse

$$\frac{a^n}{\mathfrak{A}}+\frac{b^n}{\mathfrak{B}}+\frac{c^n}{\mathfrak{C}}+\frac{d^n}{\mathfrak{D}}+\ldots\ldots+\frac{z^n}{\mathfrak{Z}}=0\,.$$

dum sit $n+1$ minor numero terminorum. Sumto ergo $n=0$ oritur casus Theorematis.

Corollarium 1.

1170. Haec si transferantur ad numeros $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, etc. supra (1160.) definitos, ubi aliquod leve discrimen in factorum constitutione probe est notandum, intelligetur esse

$$\frac{1}{\mathfrak{A}}+\frac{1}{\mathfrak{B}}+\frac{1}{\mathfrak{C}}+\frac{1}{\mathfrak{D}}+\text{etc.}=0$$
$$-\frac{\alpha}{\mathfrak{A}}-\frac{\beta}{\mathfrak{B}}-\frac{\gamma}{\mathfrak{C}}-\frac{\delta}{\mathfrak{D}}-\text{etc.}=0$$
$$+\frac{\alpha^2}{\mathfrak{A}}+\frac{\beta^2}{\mathfrak{B}}+\frac{\gamma^2}{\mathfrak{C}}+\frac{\delta^2}{\mathfrak{D}}+\text{etc.}=0$$
$$-\frac{\alpha^3}{\mathfrak{A}}-\frac{\beta^3}{\mathfrak{B}}-\frac{\gamma^3}{\mathfrak{C}}-\frac{\delta^3}{\mathfrak{D}}-\text{etc.}=0$$
etc.

donec perveniatur ad hanc formam

$$\pm\frac{\alpha^{n-1}}{\mathfrak{A}}\pm\frac{\beta^{n-1}}{\mathfrak{B}}\pm\frac{\gamma^{n-1}}{\mathfrak{C}}\pm\frac{\delta^{n-1}}{\mathfrak{D}}\pm\text{etc.}$$

cujus summa non amplius est evanescens; sed aequalis fractioni $\frac{1}{N}$.

Corollarium 2.

1171. Hoc etiam ex evolutione formae in Theoremate adhibitae colligere licet; etenim si ea statuatur

$$\frac{z^{n-1}}{(z-a)(z-b)(z-c)\ldots\ldots(z-y)},$$

existente omnium litterarum a, b, c, etc. numero $= n$, quia hic numerator z^{n-1} tot habet dimensiones, quot sunt factores in denominatore, pars integra in hac fractione contenta est unitas; quae etiam facta resolutione conservatur, et in applicatione ad casum memoratum abit in $\frac{1}{N}$.

Scholion.

1172. Post hujus Theorematis demonstrationem demum clare a posteriori ostendi potest, quemadmodum integrale supra (1160.) exhibitum aequationi differentiali ibidem propositae satisfaciat. Notatis enim expressionibus §. 1170, cum supra invenerimus integrale

$$y = \frac{1}{\mathfrak{A}} e^{-\alpha x} \int e^{\alpha x} \mathrm{X}\, \partial x + \frac{1}{\mathfrak{B}} e^{-\beta x} \int e^{\beta x} \mathrm{X}\, \partial x + \text{etc.}$$

erit continuo differentiando

$$\frac{\partial y}{\partial x} = -\frac{\alpha}{\mathfrak{A}} e^{-\alpha x} \int e^{\alpha x} \mathrm{X}\, \partial x - \frac{\beta}{\mathfrak{B}} e^{-\beta x} \int e^{\beta x} \mathrm{X}\, \partial x - \text{etc.}$$

$$\frac{\partial\partial y}{\partial x^2} = \frac{\alpha^2}{\mathfrak{A}} e^{-\alpha x} \int e^{\alpha x} \mathrm{X}\, \partial x + \frac{\beta^2}{\mathfrak{B}} e^{-\beta x} \int e^{\beta x} \mathrm{X}\, \partial x + \text{etc.}$$

$$\frac{\partial^3 y}{\partial x^3} = \frac{-\alpha^3}{\mathfrak{A}} e^{-\alpha x} \int e^{\alpha x} \mathrm{X}\, \partial x - \frac{\beta^3}{\mathfrak{B}} e^{-\beta x} \int e^{\beta x} \mathrm{X}\, \partial x - \text{etc.}$$

etc.

usque ad

$$\frac{\partial^{n-1} y}{\partial x^{n-1}} = \pm \frac{\alpha^{n-1}}{\mathfrak{A}} e^{-\alpha x} \int e^{\alpha x} \mathrm{X}\, \partial x \pm \frac{\beta^{n-1}}{\mathfrak{B}} e^{-\beta x} \int e^{\beta x} \mathrm{X}\, \partial x \pm \text{etc.}$$

unde sequens forma differentialis resultat

$$\frac{\partial^n y}{\partial x^n} = \mp \frac{\alpha^n}{\mathfrak{A}} e^{-\alpha x} \int e^{\alpha x} X \partial x \mp \frac{\beta^n}{\mathfrak{B}} e^{-\beta x} \int e^{\beta x} X \partial x \mp \text{etc.}$$

$$\mp \left(\frac{\alpha^{n-1}}{\mathfrak{A}} + \frac{\beta^{n-1}}{\mathfrak{B}} + \frac{\gamma^{n-1}}{\mathfrak{C}} + \text{etc.} \right) X,$$

quod postremum membrum abit in $\frac{1}{N}$ X.

Si jam omnes hae formae singulatim multiplicentur per quantitates A, B, C, D . . . . . N, quoniam est

$$A - B\alpha + C\alpha^2 - D\alpha^3 + \ldots\ldots \mp N\alpha^n = 0,$$
$$A - B\beta + C\beta^2 - D\beta^3 + \ldots\ldots \mp N\beta^n = 0,$$

propterea quod $\alpha + z$, $\beta + z$, $\gamma + z$, etc. sunt factores formae

$$A + Bz + Cz^2 + Dz^3 + \ldots\ldots + Nz^n,$$

manifesto obtinebimus

$$Ay + B\frac{\partial y}{\partial x} + C\frac{\partial \partial y}{\partial x^2} + D\frac{\partial^3 y}{\partial x^3} + \ldots\ldots + N\frac{\partial^n y}{\partial x^n} = X,$$

quae est ipsa aequatio differentialis initio proposita.

Problema 153.

1173. Proposita aequatione differentiali cujuscunque gradus

$$X = Ay + B\frac{\partial y}{\partial x} + C\frac{\partial \partial y}{\partial x^2} + D\frac{\partial^3 y}{\partial x^3} + \ldots\ldots + N\frac{\partial^n y}{\partial x^n},$$

si expressio algebraica hinc formata

$$P = A + Bz + Cz^2 + Dz^3 \ldots\ldots\ldots + Nz^n$$

duos habeat factores simplices imaginarios factore duplici $ff + 2fz\cos.\theta + zz$ contentos, investigare partes integralis hinc oriundas.

Solutio.

Sint $\alpha + z$ et $\beta + z$ hi duo factores imaginarii, ut sit $\alpha = f(\cos.\,\theta + \sqrt{-1}.\sin.\,\theta)$ et $\beta = f(\cos.\,\theta - \sqrt{-1}.\sin.\,\theta)$, ob

$$(\alpha + z)(\beta + z) = ff + 2fz\cos.\,\theta + zz,$$

ac statuatur $P = (ff + 2fz\cos.\,\theta + zz)\,Q$, existente

$$Q = A' + B'z + C'z^2 + \ldots\ldots + N'z^{n-2}.$$

Cum igitur integralis partes ex binis illis factoribus simplicibus imaginariis ortae sint

$$\frac{1}{\mathfrak{A}} e^{-\alpha x} \int e^{\alpha x} X \partial x + \frac{1}{\mathfrak{B}} e^{-\beta x} \int e^{\beta x} X \partial x = v,$$

hos valores imaginarios ad realitatem perduci oportet. Erunt autem $\mathfrak{A}$ et $\mathfrak{B}$ quantitates imaginariae resultantes ex forma

$$(f\cos.\,\theta \mp \sqrt{-1}.f\sin.\,\theta + z)\,Q,$$

si loco z scribatur

$$-f\cos.\,\theta \mp \sqrt{-1}.f\sin\,\theta.$$

At facta hac substitutione fit

$$Q = A' - B'f\cos.\,\theta + C'ff\cos.\,2\theta - D'f^3\cos.\,3\theta + \text{etc.}$$
$$\mp\sqrt{-1}.B'f\sin.\,\theta \pm \sqrt{-1}.C'ff\sin.\,2\theta \mp \sqrt{-1}.D'f^3\cos.\,3\theta \pm \text{etc.}$$

Ponamus brevitatis gratia

$$A' - B'f\cos.\,\theta + C'ff\cos.\,2\theta - D'f^3\cos.\,3\theta + \text{etc.} = \mathfrak{M} \text{ et}$$
$$-B'f\sin.\,\theta + C'ff\sin.\,2\theta - D'f^3\sin.\,3\theta + \text{etc.} = \mathfrak{N},$$

ut sit $Q = \mathfrak{M} \pm \mathfrak{N}\sqrt{-1}$, ubi signorum ambiguorum superius valet pro litteris α et $\mathfrak{A}$, inferius pro litteris β et $\mathfrak{B}$. Hinc ergo erit

$$\mathfrak{A} = -2\sqrt{-1}.f\sin.\,\theta\,(\mathfrak{M} + \mathfrak{N}\sqrt{-1}) \text{ et}$$
$$\mathfrak{B} = +2\sqrt{-1}.f\sin.\,\theta\,(\mathfrak{M} - \mathfrak{N}\sqrt{-1}),$$

ideoque

$$2v\sqrt{-1}\,.\,f\sin\theta = \frac{-e^{-\alpha x}\int e^{\alpha x}X\partial x}{\mathfrak{M}+\mathfrak{N}\sqrt{-1}} + \frac{e^{-\beta x}\int e^{\beta x}X\partial x}{\mathfrak{M}-\mathfrak{N}\sqrt{-1}}.$$

Est vero

$$e^{\alpha x} = e^{fx\cos.\theta}[\cos.(fx\sin.\theta)+\sqrt{-1}\,.\,\sin.(fx\sin.\theta)] \text{ et}$$
$$e^{\beta x} = e^{fx\cos.\theta}[\cos.(fx\sin.\theta)-\sqrt{-1}\,.\,\sin.(fx\sin.\theta)].$$

Sit brevitatis gratia angulus $fx\sin.\theta=\Phi$ erit

$$2\sqrt{-1}\,.\,v(\mathfrak{M}\mathfrak{M}+\mathfrak{N}\mathfrak{N})f\sin.\theta =$$
$$-(\mathfrak{M}-\mathfrak{N}\sqrt{-1})e^{-fx\cos.\theta}(\cos.\Phi-\sqrt{-1}\,.\sin.\Phi)\int e^{fx\cos.\theta}X\partial x(\cos.\Phi+\sqrt{-1}\,.\sin.\Phi)$$
$$+(\mathfrak{M}+\mathfrak{N}\sqrt{-1})e^{-fx\cos.\theta}(\cos.\Phi+\sqrt{-1}\,.\sin.\Phi)\int e^{fx\cos.\theta}X\partial x(\cos.\Phi-\sqrt{-1}\,.\sin.\Phi)$$
$$= e^{-fx\cos.\theta}2\sqrt{-1}\,.(\mathfrak{M}\sin.\Phi+\mathfrak{N}\cos.\Phi)\int e^{fx\cos.\theta}X\partial x\cos.\Phi$$
$$- e^{-fx\cos.\theta}2\sqrt{-1}\,.(\mathfrak{M}\cos.\Phi-\mathfrak{N}\sin.\Phi)\int e^{fx\cos.\theta}X\partial x\sin.\Phi.$$

Quocirca habebimus integralis partem quaesitam

$$v = \frac{\left\{\begin{array}{l}+e^{-fx\cos.\theta}(\mathfrak{M}\sin\Phi+\mathfrak{N}\cos.\Phi)\int e^{fx\cos.\theta}X\partial x\cos.\Phi\\ -e^{-fx\cos.\theta}(\mathfrak{M}\cos.\Phi-\mathfrak{N}\sin.\Phi)\int e^{fx\cos.\theta}X\partial x\sin.\Phi\end{array}\right\}}{(\mathfrak{M}\mathfrak{M}+\mathfrak{N}\mathfrak{N})f\sin.\theta}$$

existente $\Phi = fx\sin.\theta$.

Corollarium 1.

1174. Praecipuum igitur opus hic consistit in inventione formulae imaginariae $\mathfrak{M}+\mathfrak{N}\sqrt{-1}$, quae colligi debet ex quantitate Q, dum loco z scribitur valor imaginarius

$$-f(\cos.\theta+\sqrt{-1}\,.\,\sin.\theta),$$

unde hoc commodi nascitur, ut loco z^n scribi oporteat

$$(-f)^n(\cos.n\theta+\sqrt{-1}\,.\,\sin.n\theta).$$

Corollarium 2.

1175. Cum sit $Q=\frac{P}{ff+2fz\cos.\theta+zz}$, etiam ex hac forma per eandem substitutionem formula imaginaria $\mathfrak{M}+\mathfrak{N}\sqrt{-1}$

inveniri potest, ubi autem notandum est, hac substitutione tam numeratorem P quam denominatorem evanescere. Ex quo manifestum est, valorem istius formulae rite obtineri ex hac fractione

$$\frac{\partial P}{2(f\cos.\theta+z)\partial z}=\frac{\partial P}{-2\sqrt{-1}.f\sin.\theta\,\partial z}.$$

Corollarium 3.

1176. Quoniam igitur est

$$\frac{\partial P}{\partial z}=B+2Cz+3Dz^2+4Ez^3+\ldots nNz^{n-1},$$

si statuamus

$$\mathfrak{P}=B-2Cf\cos.\theta+3Df^2\cos.2\theta-4Ef^3\cos.3\theta \ldots \pm nNf^{n-1}\cos.(n-1)\theta,$$

$$\mathfrak{Q}=-2Cf\sin\theta+3Df^2\sin.2\theta-4Ef^3\sin.3\theta\ldots \pm nNf^{n-1}\sin.(n-1)\theta,$$

ut facta substitutione fiat

$$\frac{\partial P}{\partial z}=\mathfrak{P}+\sqrt{-1}.\mathfrak{Q},$$

habebimus

$$\mathfrak{M}+\sqrt{-1}.\mathfrak{N}=\frac{\mathfrak{P}+\sqrt{-1}.\mathfrak{Q}}{-2\sqrt{-1}.f\sin.\theta}=\frac{-\mathfrak{Q}+\sqrt{-1}.\mathfrak{P}}{2f\sin.\theta},$$

ideoque

$$\mathfrak{M}=\frac{-\mathfrak{Q}}{2.f\sin.\theta}\ \text{et}\ \mathfrak{N}=\frac{\mathfrak{P}}{2f\sin.\theta}.$$

Corollarium 4.

1177. Immediate ergo ex quantitate P indeque derivatis $\mathfrak{P}$ et $\mathfrak{Q}$, posito $fx\sin.\theta=\Phi$, integralis pars ex factore duplici $ff+2fz\cos.\theta+zz$ nata erit expressa

$$v=\frac{2e^{-fx\cos.\theta}}{\mathfrak{P}\mathfrak{P}+\mathfrak{Q}\mathfrak{Q}}\left\{\begin{array}{l}(\mathfrak{P}\cos.\Phi-\mathfrak{Q}\sin.\Phi)\int e^{fx\cos.\theta}X\partial x\cos.\Phi\\+(\mathfrak{P}\sin.\Phi+\mathfrak{Q}\cos.\Phi)\int e^{fx\cos.\theta}X\partial x\sin.\Phi)\end{array}\right\}$$

Scholion.

1178. Quotcunque ergo forma

$$P = A + Bz + Cz^2 + Dz^3 + \text{etc.}$$

habuerit factores duplices, pro singulis ope horum praeceptorum partes integralis facile definiuntur, et quia hinc inventio partium, quae factoribus simplicibus conveniunt, sive ii sint inaequales, non turbatur, omnibus partibus in unam summam conjectis habebitur integrale completum aequationis differentialis propositae. Verum tamen haec praecepta non sufficiunt, si factorum duplicium bini pluresve inter se fuerint aequales; hujusmodi enim casus peculiarem exigunt evolutionem similem ejus, qua pro casu duorum pluriumve factorum simplicium inter se aequalium sum usus. Ne autem hanc tractionem nimis protraham, sufficiet casum pro duobus factoribus duplicibus inter se aequalibus evolvisse, cum inde methodus ad plures facile extendatur.

Problema 154.

1179. Proposita aequatione differentiali cujuscunque gradus

$$X = Ay + B\frac{\partial y}{\partial x} + C\frac{\partial\partial y}{\partial x^2} + D\frac{\partial^3 y}{\partial x^3} + \ldots\ldots + N\frac{\partial^n y}{\partial x^n},$$

si expressio algebraica inde formata

$$P = A + Bz + Cz^2 + Dz^3 + \ldots + Nz^n$$

habeat factorem duplicem quadratum

$$(ff + 2fz\cos.\theta + zz)^2,$$

partem integralis ei convenientem investigare.

Solutio.

Ponamus ergo $P = (ff + 2fz\cos.\theta + zz)^2 Q$, sitque

$$Q = A' + B'z + C'zz + \ldots\ldots + N'z^{n-4},$$

ac primo imaginaria non curantes statuamus

$$\alpha = f(\cos.\theta + \sqrt{-1}.\sin.\theta) \text{ et}$$
$$\beta = f(\cos.\theta - \sqrt{-1}.\sin.\theta),$$

ut sit

$$P = (\alpha + z)^2 (\beta + z)^2 Q.$$

Jam ex iis quae supra (1163.) de binis factoribus simplicibus aequalibus docuimus, ponamus formam

$$\frac{P}{(\alpha + z)^2} = (\beta + z)^2 Q, \text{ posito } z = -\alpha, \text{ abire in } \mathfrak{A},$$

at hanc formam

$$\frac{P}{(\beta + z)^2} = (\alpha + z)^2 Q, \text{ posito } z = -\beta, \text{ in } \mathfrak{B},$$

quibus quantitatibus $\mathfrak{A}$ et $\mathfrak{B}$ inventis, ibi ostendi fore integralis partes hinc oriundas

$$\frac{1}{\mathfrak{A}} e^{-\alpha x} \int \partial x \int e^{\alpha x} X \partial x + \frac{1}{\mathfrak{B}} e^{-\beta x} \int \partial x \int e^{\beta x} X \partial x = v,$$

quas, cum jam imaginaria involvant, ad realitatem reduci oportet. Faciamus ut in problemate praecedente

$$\mathfrak{M} = A' - B'f\cos.\theta + C'f^2\cos.2\theta - D'f^3\cos.3\theta + \text{etc.}$$
$$\mathfrak{N} = \quad - B'f\sin.\theta + C'f^2\sin.2\theta - D'f^3\sin.3\theta + \text{etc.}$$

ut quantitas Q, posito $z = -\alpha = -f(\cos.\theta + \sqrt{-1}.\sin.\theta)$, abeat in $\mathfrak{M} + \mathfrak{N}\sqrt{-1}$, at posito

$$z = -\beta = -f(\cos.\theta - \sqrt{-1}.\sin.\theta),$$

in $\mathfrak{M} - \mathfrak{N}\sqrt{-1}$.

Cum jam sit $(\beta - \alpha)^2 = (-2\sqrt{-1}.f\sin.\theta)^2 = -4ff\sin.\theta^2$, cui quoque $(\alpha - \beta)^2$ aequatur, erit

$$\mathfrak{A} = -4ff\sin.\theta^2 (\mathfrak{M} + \mathfrak{N}\sqrt{-1}) \text{ et}$$
$$\mathfrak{B} = -4ff\sin.\theta^2 (\mathfrak{M} - \mathfrak{N}\sqrt{-1}),$$

ideoque

$$-4ff\sin.\theta^2(\mathfrak{M}\mathfrak{M}+\mathfrak{N}\mathfrak{N})v=(\mathfrak{M}-\mathfrak{N}\surd-1)e^{-\alpha x}\int\partial x\int e^{\alpha x}\mathrm{X}\partial x$$
$$+(\mathfrak{M}+\mathfrak{N}\surd-1)e^{-\beta x}\int\partial x\int e^{\beta x}\mathrm{X}\partial x.$$

At posito $fx\sin.\theta=\Phi$, est ut vidimus

$$e^{\alpha x}=e^{fx\cos.\theta}(\cos.\Phi+\surd-1.\sin.\Phi),$$
$$e^{-\alpha x}=e^{-fx\cos.\theta}(\cos.\Phi-\surd-1.\sin.\Phi),$$
$$e^{\beta x}=e^{fx\cos.\theta}(\cos.\Phi-\surd-1.\sin.\Phi),$$
$$e^{-\beta x}=e^{-fx\cos.\theta}(\cos.\Phi+\surd-1.\sin.\Phi),$$

unde illius aequationis alterum membrum abit in

$$+e^{-fx\cos.\theta}[\mathfrak{M}\cos.\Phi-\mathfrak{N}\sin.\Phi-\mathfrak{N}\surd-1.\cos.\Phi-\mathfrak{M}\surd-1.\sin.\Phi]$$
$$\times\int\partial x\int e^{fx\cos.\theta}\mathrm{X}\partial x(\cos.\Phi+\surd-1.\sin.\Phi)$$
$$+e^{-fx\cos.\theta}[\mathfrak{M}\cos.\Phi-\mathfrak{N}\sin.\Phi+\mathfrak{N}\surd-1.\cos.\Phi+\mathfrak{M}\surd-1.\sin.\Phi]$$
$$\times\int\partial x\int e^{fx\cos.\theta}\mathrm{X}\partial x(\cos.\Phi-\surd-1.\sin.\Phi),$$

ubi partes imaginariae sponte se destruunt, ita ut obtineatur

$$v=\frac{\left\{\begin{array}{l}-e^{-fx\cos.\theta}(\mathfrak{M}\cos.\Phi-\mathfrak{N}\sin.\Phi)\int\partial x\int e^{fx\cos.\theta}\mathrm{X}\partial x\cos.\Phi\\-e^{-fx\cos.\theta}(\mathfrak{N}\cos.\Phi+\mathfrak{M}\sin.\Phi)\int\partial x\int e^{fx\cos.\theta}\mathrm{X}\partial x\sin.\Phi\end{array}\right\}}{2(\mathfrak{M}\mathfrak{M}+\mathfrak{N}\mathfrak{N})ff\sin.\theta^2}$$

seu hoc modo

$$v=\frac{-e^{-fx\cos.\theta}}{2(\mathfrak{M}\mathfrak{M}+\mathfrak{N}\mathfrak{N})ff\sin.\theta^2}\left\{\begin{array}{l}+(\mathfrak{M}\cos.\Phi-\mathfrak{N}\sin.\Phi)\int\partial x\int e^{fx\cos.\theta}\mathrm{X}\partial x\cos.\Phi\\+(\mathfrak{M}\sin.\Phi+\mathfrak{N}\cos.\Phi)\int\partial x\int e^{fx\cos.\theta}\mathrm{X}\partial x\sin\Phi\end{array}\right\},$$

quae expressio ab imaginariis penitus est liberata.

Nota. Etiam haec solutio insigni correctione indiget diligentiae lectorum relicta.

Corollarium 1.

1180. Quoniam formula imaginaria $\mathfrak{M}+\mathfrak{N}\surd-1$ nascitur ex quantitate Q, si loco z scribatur

$$= -f(\cos.\theta + \sqrt{-1}.\sin.\theta),$$

eadem positione quoque reperietur ex forma

$$\frac{P}{(ff + 2fz\cos.\theta + zz)^2},$$

verum hic tam numerator quam denominator prodit evanescens.

Corollarium 2.

1181. Orietur ergo quoque idem valor ex formula

$$\frac{\partial P}{4\partial z[f^3\cos.\theta + ffz(1 + 2\cos.\theta^2) + 3fzz\cos.\theta + z^3]},$$

ubi cum idem incommodum denuo recurrat, orietur quoque ex hac formula

$$\frac{\partial\partial P}{4\partial z^2[ff(1 + 2\cos.\theta^2) + 6fz\cos.\theta + 3zz]}.$$

Corollarium 3.

1182. Statuatur hic primo in denominatore

$$z = -f(\cos.\theta + \sqrt{-1}.\sin.\theta)$$

fietque haec formula

$$\frac{-\partial\partial P}{8ff\partial z^2\sin.\theta^2}.$$

Deinde cum sit

$$\frac{\partial\partial P}{2\partial z^2} = C + 3Dz + 6Ezz + \ldots\ldots + \frac{n(n-1)}{1.2}Nz^{n-2},$$

ponamus brevitatis gratia

$$\mathfrak{P} = C - 3Df\cos.\theta + 6Eff\cos.2\theta - \ldots\ldots \pm \frac{n(n-1)}{1.2}Nf^{n-2}\cos.(n-2)\theta,$$

$$\mathfrak{Q} = -3Df\sin.\theta + 6Eff\sin.2\theta - \ldots\ldots \pm \frac{n(n-1)}{1.2}Nf^{n-2}\sin.(n-2)\theta,$$

ut sit facta substitutione

$$\frac{\partial\partial P}{2\partial z^2} = \mathfrak{P} + \mathfrak{Q}\sqrt{-1},$$

ideoque

$$\mathfrak{M} + \mathfrak{N}\sqrt{-1} = \frac{-\mathfrak{P} - \mathfrak{Q}\sqrt{-1}}{4ff\sin.\theta^2},$$

et consequenter

$$\mathfrak{M} = \frac{-\mathfrak{P}}{4ff\sin.\theta^2} \text{ et } \mathfrak{N} = \frac{-\mathfrak{Q}}{4ff\sin.\theta^2}.$$

Quos ergo valores in parte integralis inventa substituere licet.

Corollarium 4.

1183. Facta autem substitutione, factor duplex quadratus $(ff + 2fz\cos.\theta + zz)^2$ hanc praebet integralis partem

$$v = \frac{2e^{-fx\cos.\theta}}{\mathfrak{P}\mathfrak{P} + \mathfrak{Q}\mathfrak{Q}} \left\{ \begin{array}{l} +(\mathfrak{P}\cos.\Phi - \mathfrak{Q}\sin.\Phi)\int\partial x\int e^{fx\cos.\theta}X\partial x\cos.\Phi \\ +(\mathfrak{P}\sin.\Phi + \mathfrak{Q}\cos.\Phi)\int\partial x\int e^{fx\cos.\theta}X\partial x\sin.\Phi \end{array} \right\},$$

ubi Φ denotat angulum $fx\sin.\theta$.

Scholion.

1184. Si hanc expressionem cum ea, quam problemate praecedente invenimus, comparemus, vix actuali simili evolutione erit opus pro casibus magis complicatis. Ita si quantitas

$$P = A + Bz + Cz^2 + Dz^3 + \ldots\ldots + Nz^n,$$

factorem habeat duplicem cubicum

$$(ff + 2fz\cos.\theta + zz)^3,$$

quantitates $\mathfrak{P}$ et $\mathfrak{Q}$ ita definiuntur, ut sit

$$\mathfrak{P} = D - 4Ef\cos.\theta + 10Fff\cos.2\theta - 20Gf^3\cos.3\theta + \ldots \pm \frac{n(n-1)(n-2)}{1.2.3}Nf^{n-3}\cos.(n-3)\theta,$$

$$\mathfrak{Q} = -4Ef\sin.\theta + 10Fff\sin.2\theta - 20Gf^3\sin.3\theta + \ldots \pm \frac{n(n-1)(n-2)}{1.2.3}Nf^{n-3}\sin.(n-3)\theta,$$

quibus inventis, erit integralis pars hinc nata

$$\frac{2e^{-fx\cos.\theta}}{\mathfrak{P}\mathfrak{P} + \mathfrak{Q}\mathfrak{Q}} \left\{ \begin{array}{l} +(\mathfrak{P}\cos.\Phi - \mathfrak{Q}\sin.\Phi)\int\partial x\int\partial x\int e^{fx\cos.\theta}X\partial x\cos.\Phi \\ +(\mathfrak{P}\sin.\Phi + \mathfrak{Q}\cos.\Phi)\int\partial x\int\partial x\int e^{fx\cos.\theta}X\partial x\sin.\Phi \end{array} \right\},$$

neque jam ulterior progressio ulli amplius difficultati est obnoxia. Quocirca aequationis hoc capite propositae resolutionem ita concinne

mihi equidem absolvisse videor, ut nihil amplius desiderari possit. Interim hoc argumentum maxime illustrabitur, si haec praecepta ad exempla particularia accommodabimus; cui instituto sequens caput est destinatum. Ante autem insignem proprietatem circa hujusmodi aequationes generales proponam, quae in Analysi ingentem usum habitura videtur.

Problema 155.

1185. Proposita aequatione differentiali cujuscunque gradus

$$X = Ay + B\frac{\partial y}{\partial x} + C\frac{\partial\partial y}{\partial x^2} + D\frac{\partial^3 y}{\partial x^3} + \ldots\ldots + N\frac{\partial^{m+n} y}{\partial x^{m+n}},$$

si formula algebraica inde nata

$$P = A + Bz + Cz^2 + Dz^3 + \ldots\ldots + Nz^{m+n}$$

duobus factoribus constet $P = QR$, ut sit

$$Q = \mathfrak{A} + \mathfrak{B}z + \mathfrak{C}z^2 + \ldots\ldots + \mathfrak{N}z^m \text{ et}$$
$$R = \mathfrak{a} + \mathfrak{b}z + \mathfrak{c}z^2 + \ldots\ldots + \mathfrak{n}z^n,$$

integrationem illius aequationis ad integrationem binarum aequationum simpliciorum revocare.

Solutio.

Si formam integralem primo (1158.) perpendamus, haud difficulter inde colligimus, postquam hanc aequationem integraverimus

$$X = \mathfrak{A}v + \mathfrak{B}\frac{\partial v}{\partial x} + \mathfrak{C}\frac{\partial\partial v}{\partial x^2} + \ldots\ldots + \mathfrak{N}\frac{\partial^m v}{\partial x^m},$$

indeque valorem ipsius v per x et X definiverimus, valorem ipsius y pro aequatione proposita ex hac aequatione erutum iri

$$v = \mathfrak{a}y + \mathfrak{b}\frac{\partial y}{\partial x} + \mathfrak{c}\frac{\partial\partial y}{\partial x^2} + \ldots\ldots + \mathfrak{n}\frac{\partial^n y}{\partial x^n},$$

cujus ratio adeo in promtu est posita; dum ex hac aequatione valores pro v ejusque differentialibus substituantur. Prodibit enim

$$\begin{aligned} X = \mathfrak{A}\mathfrak{a}y &+ \mathfrak{A}\mathfrak{b}\cdot\frac{\partial y}{\partial x} + \mathfrak{A}\mathfrak{c}\cdot\frac{\partial\partial y}{\partial x^2} + \mathfrak{A}\mathfrak{d}\cdot\frac{\partial^3 y}{\partial x^3} + \text{etc.} \\ &+ \mathfrak{B}\mathfrak{a} \quad\;\; + \mathfrak{B}\mathfrak{b} \qquad\;\; + \mathfrak{B}\mathfrak{c} \\ &\qquad\qquad\; + \mathfrak{C}\mathfrak{a} \qquad\;\; + \mathfrak{C}\mathfrak{b} \\ &\qquad\qquad\qquad\qquad\quad\;\; + \mathfrak{D}\mathfrak{a} \end{aligned}$$

Cum autem per hypothesin sit $P = QR$, seriebus Q et R in se multiplicatis, necesse est fieri

$$A = \mathfrak{A}\mathfrak{a},\; B = \mathfrak{A}\mathfrak{b} + \mathfrak{B}\mathfrak{a},\; C = \mathfrak{A}\mathfrak{c} + \mathfrak{B}\mathfrak{b} + \mathfrak{C}\mathfrak{a}, \text{ etc.}$$

sicque haec postrema aequatio ad ipsam propositam reducitur.

Corollarium 1.

1186. Si tantum ad factores simplices respiciamus, prioris aequationis integrale per hujusmodi terminos exprimitur

$$v = \Gamma e^{-\alpha x}\int e^{\alpha x} X\partial x \text{ etc.}$$

posterioris vero aequationis integrale per hujusmodi

$$y = \Delta e^{-\beta x}\int e^{\beta x} v\partial x \text{ etc.}$$

Corollarium 2.

1187. Quodsi jam in singulis terminis posterioris integralis substituamus singulos prioris, fiet

$$y = \Gamma\Delta e^{-\beta x}\int e^{(\beta-\alpha)x}\partial x\int e^{\alpha x} X\partial x,$$

quae forma ad hanc reducitur

$$y = \frac{\Gamma\Delta}{\beta-\alpha}\left(e^{-\alpha x}\int e^{\alpha x} X\partial x - e^{-\beta x}\int e^{\beta x} X\partial x\right),$$

cujusmodi termini per integrationem aequationis propositae immediate inveniuntur.

Corollarium 3.

1188. Si hic fuisset $\beta = \alpha$, sine ulla reductione statim prodiisset forma

$$y = \Gamma \Delta e^{-\alpha x} \int \partial x \int e^{\alpha x} X \partial x$$

supra pro casu duorum factorum simplicium aequalium inventa. Interim cum totum negotium ad resolutionem in factores vel simplices vel duplices reales redeat, ipsa aequatio proposita modo exposito facillime expeditur.

CAPUT IV.

APPLICATIO METHODI INTEGRANDI IN CAPITE PRAECEDENTE TRADITAE AD EXEMPLA.

Problema 156.

1189. Proposita hac aequatione differentiali

$$X = a^n y + \frac{\partial^n y}{\partial x^n}$$

ejus integrale completum invenire.

Solutio.

Hic ergo est $P = a^n + z^n$ ubi primo observetur, si n sit numerus impar, factorem simplicem esse $a + z$, ex quo nascitur pars integralis

$$\frac{1}{\mathfrak{A}} e^{-ax} \int e^{ax} X \partial x,$$

existente $\mathfrak{A}$ valore ex forma $\frac{P}{a+z}$ emergente, si ponatur $z = -a$, qui ergo valor cum sit etiam $\frac{\partial P}{\partial z} = n z^{n-1}$, ob $n - 1$ numerum parem, erit $\mathfrak{A} = n a^{n-1}$, ideoque haec integralis pars

$$\frac{1}{n a^{n-1}} e^{-ax} \int e^{ax} X \partial x.$$

Reliqui factores omnes in hac forma continentur

$$aa - 2az \cos. \phi + zz, \text{ existente } \phi = \frac{(2i+1)\pi}{n},$$

ubi i denotat numerum integrum quemcunque et π angulum duobus rectis aequalem. Comparata hac forma cum Probl. 153. et

Coroll. 1. sit $f = -a$, et ob $x = a(\cos.\theta + \sqrt{-1}.\sin.\theta)$, ex forma $\frac{\partial P}{\partial x}$ colligitur

$$\mathfrak{P} = n a^{n-1} \cos.(n-1)\theta \quad \text{et} \quad \mathfrak{Q} = n a^{n-1} \sin.(n-1)\theta;$$

cum igitur sit

$$\cos. n\theta = -1 \text{ et } \sin. n\theta = 0, \text{ erit}$$

$$\mathfrak{P} = -n a^{n-1} \cos.\theta \text{ et } \mathfrak{Q} = n a^{n-1} \sin\theta.$$

Quare posito $f x \sin.\theta = -a x \sin.\theta = \Phi$, integralis pars ex quolibet factore duplici oriunda est

$$\frac{2 e^{a x \cos.\theta}}{n a^{n-1}} \left\{ \begin{array}{l} (-\cos.\theta \cos.\Phi - \sin.\theta \sin.\Phi) \int e^{-a x \cos.\theta} X \partial x \cos.\Phi \\ (-\cos.\theta \sin.\Phi + \sin.\theta \cos.\Phi) \int e^{-a x \cos.\theta} X \partial x \sin.\Phi \end{array} \right\}$$

seu

$$\frac{-2 e^{a x \cos.\theta}}{n a^{n-1}} [\cos.(\theta - \Phi) \int e^{-a x \cos.\theta} X \partial x \cos.\Phi - \sin.(\theta - \Phi) \int e^{-a x \cos.\theta} X \partial x \sin.\Phi]$$

et pro Φ valore restituto

$$\frac{-2 e^{a x \cos.\theta}}{n a^{n-1}} \left\{ \begin{array}{l} \cos.(\theta + a x \sin.\theta) \int e^{-a x \cos.\theta} X \partial x \cos.(a x \sin.\theta) \\ + \sin.(\theta + a x \sin.\theta) \int e^{-a x \cos.\theta} X \partial x \sin.(a x \sin.\theta). \end{array} \right\}$$

Jam pro θ successive substituantur anguli $\frac{\pi}{n}, \frac{3\pi}{n}, \frac{5\pi}{n}, \frac{7\pi}{n}$, quamdiu ipso π sunt minores, omnesque hae formae in unam summam conjectae, quibus casu quo n est numerus impar insuper addi oportet formam primo inventam

$$\frac{1}{n a^{n-1}} e^{-a x} \int e^{a x} X \partial x,$$

dabunt integrale quaesitum.

Corollarium 1.

1190. Casu quidem quo n est numerus impar, ultimus valor ipsius θ foret π, quem autem hic omitti jussimus, inde autem ob

$ax \sin.\theta = 0$ et $\cos.\theta = -1$,
prodiret ultima pars integralis

$$-\frac{2\,e^{-ax}}{n\,a^{n-1}} \int e^{ax} X \partial x,$$

dupla ejus quam capi convenit, cujus ratio est, quod sumto $\theta = \pi$ formula $aa + 2az + zz$ non amplius ipsa est factor, sed ejus radix quadrata $a + z$, ex quo hunc casum seorsim erui necesse erat.

Corollarium 2.

1191. Si est $X = 0$, formulae integrales abeunt in constantes arbitrarias, et ex factore

$$aa - 2az \cos.\theta + zz$$

oritur haec pars integralis

$$\frac{-2\,e^{ax\cos.\theta}}{n\,a^{n-1}} [\mathfrak{A} \cos.(\theta + ax \sin.\theta) + \mathfrak{B} \sin.(\theta + ax \sin.\theta)],$$

quae reducitur ad hanc formam

$$A\, e^{ax\cos.\theta} \cos.(\zeta + ax \sin.\theta),$$

denotante ζ angulum constantem quemcunque, uti jam supra invenimus.

Problema 157.

1192. Proposita hac aequatione differentiali

$$X = a^n y - \frac{\partial^n y}{\partial x^n}$$

ejus integrale completum invenire.

Solutio.

Forma algebraica hinc nata $P = a^n - z^n$ factorem semper habet $a - z$, unde nascitur pars integralis $\frac{1}{n} e^{ax} \int e^{-ax} X \partial x$,

existente $\mathfrak{A} = \frac{P}{z-a}$ posito $z = a$. Cum ergo sit quoque

$$\mathfrak{A} = \frac{\partial P}{\partial z} = -n z^{n-1}, \text{ erit } \mathfrak{A} = -n a^{n-1},$$

ideoque haec pars integralis

$$\frac{-1}{n a^{n-1}} e^{ax} \int e^{-ax} X \partial x.$$

Deinde si n sit numerus par, hincque $n-1$ impar, factor quoque erit $a+z$, qui praebet integralis partem

$$\frac{1}{n a^{n-1}} e^{-ax} \int e^{ax} X \partial x.$$

Reliqui factores omnes ipsius P sunt duplicis formae $aa - 2az \cos.\theta + zz$, existente angulo $\theta = \frac{2i\pi}{n}$, qua cum generali supra usurpata $ff + 2fz\cos.\theta + zz$ comparata, fit $f = -a$, et ex forma $\frac{\partial P}{\partial z} = -n z^{n-1}$ quaeri oportet formulam $\mathfrak{P} + \mathfrak{Q}\sqrt{-1}$, posito $z = a(\cos.\theta + \sqrt{-1}.\sin.\theta)$, unde colligitur

$$\mathfrak{P} = -n a^{n-1} \cos.(n-1)\theta \text{ et } \mathfrak{Q} = -n a^{n-1} \sin.(n-1)\theta,$$

seu ob $\cos.n\theta = 1$ et $\sin.n\theta = 0$, fit

$$\mathfrak{P} = -n a^{n-1} \cos.\theta \text{ et } \mathfrak{Q} = +n a^{n-1} \sin.\theta.$$

Posito jam angulo $-ax\sin.\theta = \Phi$, ex §. 1177. oritur pars integralis

$$\frac{2e^{ax\cos.\theta}}{n a^{n-1}} \left\{ \begin{array}{l} (-\cos.\theta\cos.\Phi + \sin.\theta\sin.\Phi) \int e^{-ax\cos.\theta} X \partial x \cos.\Phi \\ +(-\cos.\theta\sin.\Phi + \sin.\theta\cos.\Phi) \int e^{-ax\cos.\theta} X \partial x \sin.\Phi \end{array} \right\},$$

quae ut ante reducitur ad hanc formam

$$\frac{-2e^{ax\cos.\theta}}{n a^{n-1}} \left\{ \begin{array}{l} \cos.(\theta + ax\sin.\theta) \int e^{-ax\cos.\theta} X \partial x \cos.(ax\sin.\theta) \\ +\sin.(\theta + ax\sin.\theta) \int e^{-ax\cos.\theta} X \partial x \sin.(ax\sin.\theta) \end{array} \right\}.$$

Hic jam pro θ successive scribantur anguli $\frac{2\pi}{n}$, $\frac{4\pi}{n}$, $\frac{6\pi}{n}$, etc. quamdiu sunt minores quam π, haeque partes omnes cum primum

inventa atque etiam altera, si n fuerit numerus par, in unam summam collectae dabunt integrale quaesitum seu valorem ipsius y.

Corollarium.

1193. Cum factor duplex generalis $aa - 2a\cos.\phi + zz$ casibus $\phi = 0$ et $\phi = \pi$ non praebeat ipsos factores simplices reales $a - z$ et $a + z$ sed eorum quadrata, haec ratio est, cur pars integralis inde eruta prodeat dupla ejus, quam capi oportet.

Problema 158.

1194. Proposita hac aequatione differentiali

$$X = y + \frac{\partial y}{\partial x} + \frac{\partial\partial y}{\partial x^2} + \frac{\partial^3 y}{\partial x^3} + \ldots\ldots + \frac{\partial^n y}{\partial x^n},$$

ejus integrale completum investigare.

Solutio.

Forma algebraica hinc nata est

$$P = 1 + z^2 + z^3 + z^4 + \ldots\ldots + z^n,$$

cujus omnes factores scrutari oportet. Cum igitur sit $P = \frac{1 - z^{n+1}}{1 - z}$, formae $1 - z^{n+1}$ factores capi convenit, excluso $1 - z$; unde primo patet, si fuerit $n + 1$ numerus par, factorem simplicem fore $1 + z$, ex quo nascitur pars integralis $\frac{1}{\mathfrak{A}} e^{-x} \int e^x X \partial x$, existente $\mathfrak{A} = \frac{P}{1+z} = \frac{1 - z^{n+1}}{1 + zz}$, posito $z = -1$. Erit ergo quoque $\mathfrak{A} = \frac{(n+1) z^n}{2z}$, ideoque $\mathfrak{A} = \frac{1}{2}(n + 1)$, ut haec pars integralis sit $\frac{2}{n+1} e^{-x} \int e^x X \partial x$.

Factorum autem duplicium forma est $1 - 2z\cos.\theta + zz$, sumto angulo $\theta = \frac{2i\pi}{n+1}$, ita ut pro §. 1176. sit $f = -1$. Consideretur forma

$$\frac{\partial P}{\partial z} = \frac{1 - (n+1)z^n + nz^{n+1}}{(1-z)^2},$$

quae posito $z = \cos.\theta + \sqrt{-1}.\sin.\theta$, abire sumitur in $\mathfrak{P} + \mathfrak{Q}\sqrt{-1}$, sicque erit

$$\mathfrak{P}+\mathfrak{Q}\sqrt{-1} = \frac{1-(n+1)\cos.n\theta+n\cos.(n+1)\theta-(n+1)\sqrt{-1}.\sin.n\theta+n\sqrt{-1}.\sin.(n+1)\theta}{1-2\cos.\theta+\cos.2\theta-2\sqrt{-1}.\sin.\theta+\sqrt{-1}.\sin.2\theta}.$$

Cum vero sit

$$\sin.(n+1)\theta = 0 \text{ et } \cos.(n+1)\theta = 1, \text{ erit}$$

$$\sin.n\theta = -\sin.\theta \text{ et } \cos.n\theta = \cos.\theta, \text{ ideoque}$$

$$\mathfrak{P}+\mathfrak{Q}\sqrt{-1} = \frac{n+1-(n+1)\cos.\theta+(n+1)\sqrt{-1}.\sin.\theta}{-2\cos.\theta+2\cos.\theta^2-2\sqrt{-1}.\sin.\theta(1-\cos\theta)}, \text{ seu}$$

$$\mathfrak{P}+\mathfrak{Q}\sqrt{-1} = \frac{n+1}{2(1-\cos.\theta)}\cdot\frac{1-\cos.\theta+\sqrt{-1}.\sin.\theta}{-\cos.\theta-\sqrt{-1}.\sin.\theta}:$$

multiplicetur hujus fractionis numerator et denominator per $-\cos.\theta + \sqrt{-1}.\sin.\theta$ et prodibit

$$\mathfrak{P}+\mathfrak{Q}\sqrt{-1} = \frac{-(n+1)[1+\cos.\theta-2\cos.\theta^2-\sqrt{-1}.\sin.\theta(1-2\cos.\theta)]}{2(1-\cos.\theta)},$$

ita ut sit

$$\mathfrak{P} = -\tfrac{1}{2}(n+1)(1+2\cos.\theta) \text{ et } \mathfrak{Q} = \tfrac{1}{2}(n+1)\frac{\sin.\theta(1-2\cos.\theta)}{1-\cos.\theta},$$

unde fit

$$\mathfrak{P}\mathfrak{P}+\mathfrak{Q}\mathfrak{Q} = \frac{(n+1)^2}{2(1-\cos.\theta)}.$$

Tum vero posito angulo $-x\sin.\theta = \Phi$, colligitur

$$\mathfrak{P}\cos.\Phi - \mathfrak{Q}\sin.\Phi = \frac{-(n+1)[\cos.(\theta-\Phi)-\cos.(2\theta-\Phi)]}{2(1-\cos.\theta)},$$

$$\mathfrak{P}\sin.\Phi + \mathfrak{Q}\cos.\Phi = \frac{+(n+1)[\sin.(\theta-\Phi)-\sin.(2\theta-\Phi)]}{2(1-\cos.\theta)},$$

cum autem sit

$$\cos.a - \cos.b = 2\sin.\frac{a+b}{2}\sin.\frac{b-a}{2}, \text{ et}$$

$$\sin.a - \sin.b = -2\sin.\frac{b-a}{2}\cos.\frac{a+b}{2},$$

fit hinc

$$\mathfrak{P}\cos.\Phi - \mathfrak{Q}\sin.\Phi = \frac{-(n+1)\sin.\frac{1}{2}(3\theta - 2\Phi)}{2\sin.\frac{1}{2}\theta} \text{ et}$$

$$\mathfrak{P}\sin.\Phi + \mathfrak{Q}\cos.\Phi = \frac{-(n+1)\cos.\frac{1}{2}(3\theta - 2\Phi)}{2\sin.\frac{1}{2}\theta},$$

ex quo integralis pars quaesita erit

$$\frac{-4}{n+1}e^{x\cos.\theta}\sin.\tfrac{1}{2}\theta\left\{\begin{array}{l}\sin.\frac{1}{2}(3\theta+2x\sin.\theta)\int e^{-x\cos.\theta}X\partial x\cos.(x\sin.\theta)\\ -\cos.\frac{1}{2}(3\theta+2x\sin.\theta)\int e^{-x\cos.\theta}X\partial x\sin.(x\sin.\theta)\end{array}\right\}.$$

Pro θ ergo successive substituantur anguli

$$\frac{2\pi}{n+1},\ \frac{4\pi}{n+1},\ \frac{6\pi}{n+1},\ \text{etc.}$$

quamdiu sunt minores quam π, haecque partes omnes in unam summam colligantur, cui si $n+1$ sit numerus par, addatur insuper $\frac{2}{n+1}e^{-x}\int e^{x}X\partial x$, sicque obtinebitur valor ipsius y.

Corollarium 1.

1195. Si aequatio proposita in infinitum progrediatur, ut sit n numerus infinitus, anguli θ priores omnes sunt infinite parvi ideoque numero infiniti, quoad numerus par $2i$ ad $n+1$ rationem finitam habere incipiat, tum autem pro θ sequentur omnes anguli finiti in progressione arithmetica increscentes, cujus differentia est $\frac{2\pi}{n+1}$, usque ad π, quorum numerus itidem est infinitus.

Corollarium 2.

1196. Quamdiu angulus θ est infinite parvus, integralis pars ex eo oriunda hanc induit formam

$$\frac{-\theta\theta e^{x}}{n+1}\left[(3+2x)\int e^{-x}X\partial x - \int e^{-x}Xx\partial x\right],$$

quae cum per cubum infiniti sit divisa, etiam multitudo infinita hujusmodi formularum pro evanescente est habenda.

Corollarium 3.

1197. Quodsi fuerit $X = 0$, ut hujus aequationis

$$0 = y + \frac{\partial y}{\partial x} + \frac{\partial \partial y}{\partial x^2} + \frac{\partial^3 y}{\partial x^3} + \ldots\ldots + \frac{\partial^n y}{\partial x^n}$$

integrale sit investigandum, erit ejus pars quaecunque

$$e^{x \cos. \theta} [A \sin. \tfrac{1}{2} (3 \theta + 2 x \sin. \theta) + \mathfrak{A} \cos. \tfrac{1}{2} (3 \theta + 2 x \sin. \theta)],$$

seu simplicius

$$A e^{x \cos. \theta} (\cos. \zeta + x \sin. \theta).$$

Cum igitur si n sit numerus infinitus, pro θ angulus quicunque accipi queat, erit istius aequationis integrale particulare quodcunque

$$y = A e^{x \cos. \theta} (\cos. x \sin. \theta + \zeta),$$

sumendo pro ζ etiam angulum quemcunque.

Scholion.

1198. Num autem hujus aequationis differentialis in infinitum excurrentis

$$X = y + \frac{\partial y}{\partial x} + \frac{\partial \partial y}{\partial x^2} + \frac{\partial^3 y}{\partial x^3} + \frac{\partial^4 y}{\partial x^4} + \text{etc.}$$

denotante X functionem quamcunque ipsius x, integrale commodius exprimi possit, quam per partium illarum innumerabilium evanescentium summam, quaestio est altioris indaginis, neque adhuc ad hunc scopum Analyseos fines satis videntur promoti. Casibus quidem, quibus X est functio rationalis integra, puta

$$X = a + b x + c x^2 + d x^3 + e x^4 + \text{etc.}$$

res nullam habet difficultatem, cum sumto

$$y = \alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \text{etc.} + v,$$

hi coëfficientes α, β, γ, etc. semper ita definiri queant ut facta substitutione prodeat talis aequatio

$$0 = v + \frac{\partial v}{\partial x} + \frac{\partial\partial v}{\partial x^2} + \frac{\partial^3 v}{\partial x^3} + \text{etc.}$$

cui particulariter satisfacit valor

$$v = A\, e^{x \cos. \theta} \cos. (x \sin. \theta + \zeta),$$

sumtis pro ζ et θ angulis quibuscunque. Verum ex dato ejusmodi valore ipsius X invenitur

$$\alpha = a - b,\ \beta = b - 2c,\ \gamma = c - 3d,\ \delta = d - 4e,\ \varepsilon = e - 5f, \text{ etc.}$$

Verum in genere cum fiat

$$\frac{\partial X}{\partial x} = \frac{\partial y}{\partial x} + \frac{\partial\partial y}{\partial x^2} + \frac{\partial^3 y}{\partial x^3} + \text{etc.}$$

evidens est semper, posito $y = X - \frac{\partial X}{\partial x} + v$, aequationem illam transformari in hanc

$$0 = v + \frac{\partial v}{\partial x} + \frac{\partial\partial v}{\partial x^2} + \frac{\partial^3 v}{\partial x^3} + \text{etc.}$$

Corollarium.

1199. En ergo praeter expectationem integrationem completam hujus aequationis differentialis in infinitum excurrentis

$$X = y + \frac{\partial y}{\partial x} + \frac{\partial\partial y}{\partial x^2} + \frac{\partial^3 y}{\partial x^3} + \frac{\partial^4 y}{\partial x^4} + \text{etc.}$$

pro qua jam novimus esse

$$y = X - \frac{\partial X}{\partial x} + A\, e^{x \cos. \theta} \cos. (x \sin. \theta + \zeta),$$

quod postremum membrum ob angulos ζ et θ arbitrarios in infinitum multiplicari potest. Haecque forma maxime complicatae illi ex solutione oriundae aequivalere est censenda.

Problema 159.

1200. Proposita hac aequatione differentiali

$$X = y + \frac{n\partial y}{a\partial x} + \frac{n(n-1)\partial\partial y}{1.2.a^2\partial x^2} + \frac{n(n-1)(n-2)}{1.\ 2.\ \ 3a^3}\frac{\partial^3 y}{\partial x^3} + \text{etc.}$$

ubi quidem n sit numerus integer affirmativus, ut terminorum numerus sit finitus, ejus integrale completum investigare.

Solutio.

Formula algebraica hinc consideranda fit

$$P = 1 + \frac{n}{1} \cdot \frac{z}{a} + \frac{n(n-1)}{1 \cdot 2} \cdot \frac{z^2}{a^2} + \text{etc.} = \left(1 + \frac{z}{a}\right)^n,$$

quae ergo meros habet factores simplices inter se aequales $z + a$. Cum igitur sit $\frac{P}{(a+z)^n} = \frac{1}{a^n}$, ex §. 1163. statim colligitur integrale quaesitum

$$y = a^n e^{-ax} \int \partial x \int \partial x \int \partial x \; . \; . \; . \; . \; . \int e^{ax} X \partial x,$$

quoad signorum integralium numerus aequetur exponenti n. Hanc autem formam sequenti modo in integralia simplicia resolvere licet, ope reductionis generalis qua esse novimus

$$\int \partial x \int V \partial x = x \int V \partial x - \int V x \partial x,$$

unde fit

$$\int \partial x \int e^{ax} X \partial x = x \int e^{ax} X \partial x - \int e^{ax} X x \partial x,$$

$$\int \partial x \int \partial x \int e^{ax} X \partial x = \tfrac{1}{2} x^2 \int e^{ax} X \partial x - x \int e^{ax} X x \partial x + \tfrac{1}{2} \int e^{ax} X x^2 \partial x,$$

$$\int \partial x \int \partial x \int \partial x \int e^{ax} X \partial x = \frac{x^3 \int e^{ax} X \partial x - 3x^2 \int e^{ax} X x \partial x + 3x \int e^{ax} X x^2 \partial x - \int e^{ax} X x^3 \partial x}{1 \, . \; 2 \, . \; 3},$$

etc.

Cum igitur signorum integralium numerus sit $= n$, concludimus fore

$$y = \frac{a^n e^{-x}}{1 . 2 \ldots\ldots (n-1)} \Big[x^{n-1} \int e^{ax} X \partial x + \frac{(n-1)}{1} x^{n-2} \int e^{ax} X x \partial x + \frac{(n-1)(n-2)}{1 \cdot 2} x^{n-3} \int e^{ax} X x^2 \partial x - \text{etc.} \Big],$$

ubi cum singula integralia constantem arbitrariam implicent, manifestum est, hoc integrale esse completum.

Corollarium 1.

1201. Si ergo esset $X = 0$, aequationis differentialis propositae integrale completum foret

$$y = e^{-ax}(Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + Dx^{n-4} + \text{etc.} \ldots\ldots + Mx + N),$$

ubi constantium arbitrariarum A, B, C, etc. numerus utique est $= n$.

Corollarium 2.

1202. Si numerus n fuerit infinitus, simulque quantitas a capiatur infinita, ut sit $a = nc$, aequatio integranda in infinitum excurret, eritque

$$X = y + \frac{\partial y}{c\,\partial x} + \frac{\partial\partial y}{1.2\,c^2\,\partial x^2} + \frac{\partial^3 y}{1.2.3\,c^3\,\partial x^3} + \text{etc.}$$

aequatio autem integralis ad hunc casum applicata nullam lucem foeneratur.

Corollarium 3.

1203. Quaecunque autem y functio fuerit ipsius x, constat si loco x scribatur $x + \frac{1}{c}$, eam abire in

$$y + \frac{\partial y}{c\,\partial x} + \frac{\partial\partial y}{1.2\,c^2\,\partial x^2} + \frac{\partial^3 y}{1.2.3\,c^3\,\partial x^3} + \text{etc.}$$

quae cum esse debeat $= X$, vicissim patet, y aequari ei functioni ipsius x quae nascitur ex X, si ibi loco x scribatur $x - \frac{1}{c}$.

Scholion 1.

1204. Quod quo facilius appareat observo, si proposita fuerit quaecunque ejusmodi aequatio

$$X = Ay + B\frac{\partial y}{\partial x} + C\frac{\partial\partial y}{\partial x^2} + D\frac{\partial^3 y}{\partial x^3} + \text{etc.}$$

semper sine ulla integratione integrale particulare per approximationem hoc modo inveniri posse: statuatur

$$y = \alpha X + \beta\frac{\partial X}{\partial x} + \gamma\frac{\partial\partial X}{\partial x^2} + \delta\frac{\partial^3 X}{\partial x^3} + \text{etc.}$$

factaque substitutione habebitur

$$\begin{array}{llll} X = A\alpha X & + A\beta \cdot \frac{\partial X}{\partial x} & + A\gamma \cdot \frac{\partial\partial X}{\partial x^2} & + A\delta \cdot \frac{\partial^3 X}{\partial x^3} + \text{etc.} \\ & + B\alpha & + B\beta & + B\gamma \\ & & + C\alpha & + C\beta \\ & & & + D\alpha \end{array}$$

sicque coëfficientes α, β, γ, δ, etc. definiuntur, ut sit $\alpha = \frac{1}{A}$, reliqui vero

$$\beta = \frac{-B\alpha}{A} = \frac{-B}{A^2},$$

$$\gamma = \frac{-C\alpha - B\beta}{A} = \frac{-C}{A^2} + \frac{BB}{A^3},$$

$$\delta = \frac{-D\alpha - C\beta - B\gamma}{A} = \frac{-D}{A^2} + \frac{2BC}{A^3} - \frac{B^3}{A^4},$$

$$\varepsilon = \frac{-E\alpha - D\beta - C\gamma - D\delta}{A} = \frac{-E}{A^2} + \frac{2BD + C^2}{A^3} - \frac{3BBC}{A^4} + \frac{B^4}{A^5},$$

etc.

quae si accommodentur ad casum problematis, fiet

$$y = X - \frac{n\partial X}{1a\partial x} + \frac{n(n+1)\partial\partial X}{1.2a^2\partial x^2} - \frac{n(n+1)(n+2)\partial^3 X}{1.2.3a^3\partial x^3} + \text{etc.}$$

Hinc casu quo $n = \infty$ et $a = nc$ colligitur

$$y = X - \frac{\partial X}{1c\partial x} + \frac{\partial\partial X}{1.2c^2\partial x^2} - \frac{\partial^3 X}{1.2.3c^3\partial x^3} + \text{etc.}$$

quae expressio etsi in infinitum excurrens manifesto definit eam ipsius x functionem, quae nascitur ex X, si loco x scribatur $x - \frac{1}{c}$. Quodsi jam hanc novam functionem signo X' indicimus, ponamusque $y = X' + v$, aequatio Corollarii 2. abit in hanc

$$0 = v + \frac{\partial v}{\partial x} + \frac{\partial\partial v}{1.2\partial x} + \frac{\partial^3 v}{1.2.3\partial x^3} + \text{etc.}$$

cujus integrale particulare quodcunque est $v = Ae^{-ncx}x^m$ existente n numero infinito, et m numero integro positivo.

Scholion 2.

1205. Haec me deducunt ad sequentem speculationem circa serierum summationem. Sit nempe series quaecunque

$$\begin{matrix} 1 & 2 & 3 & 4 & & x \\ A, & B, & C, & D, & \ldots\ldots & T, \end{matrix}$$

cujus terminus indici x respondens sit T functio quaecunque ipsius x. Statuatur summa omnium horum terminorum

$$A + B + C + D + \ldots\ldots + T = y,$$

at perspicuum est y fore ejusmodi functionem ipsius x, ut si in ea loco x scribatur $x - 1$, proditura sit eadem illa summa y termino ultimo T mulctata, scilicet $y - T$. At loco x scribendo $x - 1$, functio y abit in

$$y - \frac{\partial y}{\partial x} + \frac{\partial \partial y}{1.2 \partial x^2} - \frac{\partial^3 y}{1.2.3 \partial x^3} + \text{etc.}$$

unde oritur haec aequatio

$$T = \frac{\partial y}{\partial x} - \frac{\partial \partial y}{1.2 \partial x^2} + \frac{\partial^3 y}{1.2.3 \partial x^3} - \frac{\partial^4 y}{1.2.3.4 \partial x^4} + \text{etc.}$$

quae semel integrata posito $\int T \partial x = X$, fit

$$X = y - \frac{\partial y}{1.2 \partial x} + \frac{\partial \partial y}{1.2.3 \partial x^2} - \frac{\partial^3 y}{1.2.3.4 \partial x^3} + \text{etc.}$$

quam quomodo integrari conveniat videamus, dum eam aliquanto generaliorem reddemus.

Problema 160.

1206. Proposita hac aequatione differentiali

$$X = \frac{ny}{a} + \frac{n(n-1)\partial y}{1.2 a^2 \partial x} + \frac{n(n-1)(n-2)\partial \partial y}{1.2.3 a^3 \partial x^2} - \text{etc.}$$

ejus integrale completum investigare.

Solutio.

Formetur inde haec quantitas algebraica

$$P = \frac{n}{a} - \frac{n(n-1)z}{1.\ 2 a^2} + \frac{n(n-1)(n-2)zz}{1.\ 2.\ 3 a^3} - \text{etc.} = \frac{1 - (1 - \frac{z}{a})^n}{z},$$

seu $P = \dfrac{a^n - (a-z)^n}{a^n z}$, cujus factor duplex quicunque hanc habebit

[illegible]

$$aa - 2a(a-z)\cos. 2\zeta + (a-z)^2,$$

existente angulo $2\zeta = \frac{2i\pi}{n}$. Abit autem haec forma in

$$2aa(1 - \cos. 2\zeta) - 2az(1 - \cos. 2\zeta) + zz,$$

vel $4aa\sin\zeta^2 - 4az\sin.\zeta^2 + zz$,

quas cum generali $ff + 2fz\cos.\theta + zz$ comparata dat

$$f = 2a\sin.\zeta, \text{ et } \cos.\theta = -\sin.\zeta,$$

unde

$$\theta = 90^0 + \zeta, \text{ et } \sin.\theta = \cos.\zeta,$$

existente $\zeta = \frac{i\pi}{n}$. Jam ad partem integralis hinc ortam inveniendam consideretur forma

$$\frac{\partial P}{\partial z} = \frac{-a^n + [a + (n-1)z](a-z)^{n-1}}{a^n zz},$$

in qua posito

$$z = -f(\cos.\theta + \sqrt{-1}.\sin.\theta), \text{ seu}$$

$$z = 2a\sin.\zeta(\sin.\zeta - \sqrt{-1}.\cos.\zeta) = a(1 - \cos. 2\zeta - \sqrt{-1}.\sin. 2\zeta),$$

ut sit

$$a - z = a(\cos. 2\zeta + \sqrt{-1}.\sin. 2\zeta);$$

prodit

$$\mathfrak{P} + \mathfrak{Q}\sqrt{-1} = \frac{-1 + [n - (n-1)(\cos. 2\zeta + \sqrt{-1}.\sin. 2\zeta)][\cos. 2(n-1)\zeta + \sqrt{-1}.\sin. 2(n-1)\zeta]}{-4aa\sin.\zeta^2(\cos. 2\zeta + \sqrt{-1}.\sin. 2\zeta)}$$

Cum autem sit

$$\cos. 2n\zeta = 1 \text{ et } \sin. 2n\zeta = 0, \text{ erit}$$

$$\cos. 2(n-1)\zeta = \cos. 2\zeta \text{ et } \sin. 2(n-1)\zeta = -\sin. 2\zeta,$$

ideoque

$$\mathfrak{P} + \mathfrak{Q}\sqrt{-1} = \frac{-n + n(\cos. 2\zeta - \sqrt{-1}.\sin. 2\zeta)}{-4aa\sin.\zeta^2(\cos. 2\zeta + \sqrt{-1}.\sin. 2\zeta)},$$

quae reducitur ad hanc formam

$$\mathfrak{P}+\mathfrak{Q}\sqrt{-1}=\frac{n}{4aa\sin.\zeta^2}(\cos.2\zeta-\sqrt{-1}.\sin.2\zeta-\cos.4\zeta+\sqrt{-1}.\sin.4\zeta),$$

unde concluditur

$$\mathfrak{P}=\frac{n}{4aa\sin.\zeta^2}(\cos.2\zeta-\cos.4\zeta)=\frac{n}{2aa\sin.\zeta}\sin.3\zeta;$$

$$\mathfrak{Q}=\frac{-n}{4aa\sin.\zeta^2}(\sin.2\zeta-\sin.4\zeta)=\frac{n}{2aa\sin.\zeta}\cos.3\zeta,$$

sicque est

$$\mathfrak{P}\mathfrak{P}+\mathfrak{Q}\mathfrak{Q}=\frac{nn}{4a^4\sin.\zeta^2},$$

et posito

$$\Phi=2ax\sin.\zeta\cos.\zeta=ax\sin.2\zeta,\ \text{fiet}$$

$$\mathfrak{P}\cos.\Phi-\mathfrak{Q}\sin.\Phi=\frac{n}{2aa\sin.\zeta}\sin.(3\zeta-\Phi)\ \text{et}$$

$$\mathfrak{P}\sin.\Phi+\mathfrak{Q}\cos.\Phi=\frac{n}{2aa\sin.\zeta}\cos.(3\zeta-\Phi).$$

Quocirca integralis pars hinc oriunda erit

$$\frac{4aa\sin.\zeta}{n}e^{2ax\sin.\zeta^2}\left\{\begin{array}{l}\sin.(3\zeta-\Phi)\int e^{-2ax\sin.\zeta^2}X\partial x\cos.\Phi\\+\cos.(3\zeta-\Phi)\int e^{-2ax\sin.\zeta^2}X\partial x\sin.\Phi\end{array}\right\}$$

ubi pro ζ successive scribi debent hi anguli

$$\frac{\pi}{n},\ \frac{2\pi}{n},\ \frac{3\pi}{n},\ \frac{4\pi}{n}\ \text{etc.}$$

quamdiu sunt angulo recto minores, at si n sit numerus par ad has partes insuper addi oportet

$$-\frac{2aa}{n}e^{2ax}\int e^{-2ax}X\partial x,$$

sicque colligetur verus valor ipsius y.

Corollarium 1.

1207. Si est $X=0$, pars integralis ex quolibet angulo $\zeta=\frac{i\pi}{n}$ nata induit hanc formam

$$e^{2ax\sin.\zeta^2}[A\sin.(3\zeta-ax\sin.2\zeta)+B\cos.(3\zeta-ax\sin.2\zeta)],$$

seu hanc

$$Ae^{2ax\sin.\zeta^2}\sin.(\alpha+ax\sin.2\zeta),$$

denotante α angulum quemcunque constantem.

Corollarium 2.

1208. Invento integrali particulari quocunque $y = V$ quod aequationi propositae satisfaciat, si ponamus deinceps $y = V + v$, orietur haec aequatio

$$0 = \frac{nv}{a} - \frac{n(n-1)\partial v}{1.2a^2\partial x} + \frac{n(n-1)(n-2)\partial\partial v}{1.2.3a^3\partial x^2} - \text{etc.}$$

ex quo integrale completum erit

$$y = V + Ae^{2ax\sin.\zeta^2}\sin.(\alpha + ax\sin.2\zeta)$$

ultima hac parte secundum omnes valores ipsius ζ multiplicata.

Corollarium 3.

1209. Si sumamus $n = \infty$ et $a = n$, ut haec prodeat aequatio differentialis in infinitum excurrens

$$X = y - \frac{\partial y}{1.2\partial x} + \frac{\partial\partial y}{1.2.3\partial x^2} - \frac{\partial^3 y}{1.2.3.4\partial x^3} + \frac{\partial^4 y}{1.2.3.4.5\partial x^4} - \text{etc.}$$

erit y terminus summatorius progressionis, cujus terminus generalis indici x respondens est $T = \frac{\partial X}{\partial x}$. Quamdiu ergo angulus $\zeta = \frac{i\pi}{n}$ est infinite parvus, ob $\Phi = 2i\pi x$, integralis pars quaelibet est

$$4i\pi e^{\frac{2ii\pi\pi}{n}x}\left\{\begin{array}{l}\sin.\left(\frac{3i\pi}{n} - 2i\pi x\right)\int e^{-\frac{2ii\pi\pi}{n}x}X\partial x\cos.(2i\pi x)\\ +\cos.\left(\frac{3i\pi}{n} - 2i\pi x\right)\int e^{-\frac{2ii\pi\pi}{n}x}X\partial x\sin.(2i\pi x)\end{array}\right\},$$

et omissis evanescentibus

$$4i\pi[\cos.(2i\pi x)\int X\partial x\sin.(2i\pi x) - \sin.(2i\pi x)\int X\partial x\cos.(2i\pi x)],$$

si jam hic pro i successive omnes numeri integri 1, 2, 3, etc. substituantur, omnium formularum hoc modo resultantium summa dabit verum et completum valorem ipsius y.

Scholion.

1210. Pro aequatione autem proposita methodo ante indicata integrale particulare per seriem differentialium invenire licet, ponendo

$$y = AX + \frac{B\partial X}{\partial x} + \frac{C\partial\partial X}{\partial x^2} + \frac{D\partial^3 X}{\partial x^3} + \frac{E\partial^4 X}{\partial x^4} + \text{etc.}$$

facta enim substitutione reperitur

$$A = \frac{a}{n},\ B = \frac{n-1}{2n},\ C = \frac{nn-1}{12an},\ D = \frac{nn-1}{24aan},$$

$$E = \frac{-(nn-1)(nn-19)}{720a^3n},\ \text{etc.}$$

cujus quidem seriei difficile est legem progressionis in genere assignare. Verum pro casu $n = \infty$ et $a = n$, qui imprimis in doctrina progressionum est notatu dignus, hi coefficientes ita se habent.

$$A = 1,\ B = \tfrac{1}{2},\ C = \tfrac{1}{12},\ D = 0,\ E = \tfrac{-1}{720},\ \text{etc.}$$

unde ea ipsa forma oritur, quam olim in genere pro termino summatorio dedi. Concessio autem hoc termino summatorio, qui fit $= V$, probe notari convenit, aequationem $y = V$ tantum esse integrale particulare aequationis propositae, completum vero facile exhiberi, si modo ad V addantur omnes hujusmodi formulae $A \sin.(\alpha + 2i\pi x)$, pro i scribendo successive omnes numeros 1, 2, 3, 4, etc. ubi pro quolibet angulus α pro arbitrio assumi potest. Quod autem singuli hi valores aequationi

$$0 = v - \frac{\partial v}{2\partial x} + \frac{\partial\partial v}{6\partial x^2} - \frac{\partial^3 v}{24\partial x^3} + \frac{\partial^4 v}{120\partial x^4} - \frac{\partial^5 v}{720\partial x^5} + \text{etc.}$$

satisfaciant, ita facillime ostenditur. Posito brevitatis gratia $2i\pi = m$, ut sit $v = \sin.(\alpha + mx)$, et facta substitutione fieri debet

$$0 = \begin{cases} \sin.(\alpha + mx)(1 - \frac{mm}{6} + \frac{m^4}{120} - \text{etc.} \\ \cos.(\alpha + mx)(-\frac{m}{2} + \frac{m^3}{24} - \frac{m^5}{720} + \text{etc.} \end{cases} = \begin{cases} \sin.(\alpha + mx)\frac{1}{m}\sin.m \\ \cos.(\alpha + mx)\frac{1}{m}(\cos.m - 1). \end{cases}$$

Cum autem sit $m = 2i\pi$, manifesto est tam $\sin.m = 0$, quam $\cos.m - 1 = 0$.

Problema 161.

1211. Proposita hac aequatione differentiali

$$X = y + \frac{n(n-1)}{1.2}\cdot\frac{\partial\partial y}{a^2\partial x^2} + \frac{n(n-1)(n-2)(n-3)}{1.2.3.4}\cdot\frac{\partial^4 y}{a^4\partial x^4} + \text{etc.}$$

ejus integrale completum investigare.

Solutio.

Quantitas algebraica hinc formanda est

$$P = 1 + \frac{n(n-1)}{1.2} \cdot \frac{zz}{aa} + \frac{n(n-1)(n-2)(n-3)}{1.\ 2.\ 3.\ 4} \cdot \frac{z^4}{a^4} + \text{etc.}$$

quae ad hanc formam manifesto reducitur

$$P = \tfrac{1}{2}(1 + \tfrac{z}{a})^n + \tfrac{1}{2} \cdot (1 - \tfrac{z}{a})^n = \frac{(a+z)^n + (a-z)^n}{2\,a^n},$$

cujus factor quicunque trinomialis est

$$(a+z)^2 - 2(aa - zz)\cos. 2\zeta + (a-z)^2,$$

sumendo

$$2\zeta = \frac{(2i+1)\pi}{n}, \text{ seu } \zeta = \frac{(2i+1)\pi}{2n}.$$

Haec autem forma abit in

$$2aa(1 - \cos. 2\zeta) + 2zz(1 + \cos. 2\zeta) = 4aa \sin. \zeta^2 + 4zz \cos. \zeta^2,$$

qui factor generalis repraesentetur hoc modo

$$aa \operatorname{tang}. \zeta^2 + zz,$$

sicque comparatio cum forma generali

$$ff + 2fz \cos. \theta + zz$$

praebet

$$f = -a \operatorname{tang}. \zeta \text{ et } \theta = 90^\circ,$$

unde fit

$$\Phi = -ax \operatorname{tang}. \zeta, \quad (1177.)$$

et valor pro z substituendus

$$-f(\cos. \theta + \sqrt{-1} . \sin. \theta) = a \operatorname{tang}. \zeta . \sqrt{-1},$$

quo pacto

$$\frac{\partial P}{\partial z} = \frac{n(a+z)^{n-1} - n(a-z)^{n-1}}{2\,a^n}.$$

abire ponitur in $\mathfrak{P}+\mathfrak{Q}\sqrt{-1}$,

unde fit

$$\mathfrak{P}+\mathfrak{Q}\sqrt{-1}=\frac{n}{2a}[(1+\text{tang.}\,\zeta.\sqrt{-1})^{n-1}-(1-\text{tang.}\,\zeta.\sqrt{-1})^{n-1}]$$

$$=\frac{n}{2a\cos.\zeta^{n-1}}[\cos.(n-1)\zeta+\sqrt{-1}.\sin.(n-1)\zeta-\cos.(n-1)\zeta+\sqrt{-1}.\sin.(n-1)\zeta],$$

ideoque $\mathfrak{P}=0$ et $\mathfrak{Q}=\frac{n\sin.(n-1)\zeta}{a\cos.\zeta^{n-1}}$. At ob $n\zeta=\frac{2i+1}{2}\pi$, hincque $\cos.n\zeta=0$ et $\sin.n\zeta=\pm 1$, prout i fuerit numerus par vel impar, erit $\sin.(n-1)\zeta=\pm\cos.\zeta$, ideoque $\mathfrak{Q}=\frac{\pm n}{a\cos.\zeta^{n-2}}$. Quocirca ob $\cos.\theta=0$, integralis pars ex hoc factore oriunda est

$$\pm\frac{2a\cos.\zeta^{n-2}}{n}(\cos.\Phi\int X\partial x\sin.\Phi-\sin.\Phi\int X\partial x\cos.\Phi),$$

seu ob $\Phi=-ax\,\text{tang.}\,\zeta$,

$$\pm\frac{2a\cos.\zeta^{n-2}}{n}\left\{\begin{array}{l}\sin.(ax\,\text{tang.}\,\zeta)\int X\partial x\cos.(ax\,\text{tang.}\,\zeta)\\-\cos.(ax\,\text{tang.}\,\zeta)\int X\partial x\sin.(ax\,\text{tang.}\,\zeta)\end{array}\right\},$$

ubi pro ζ successive substituantur anguli

$$\frac{\pi}{2n},\ \frac{3\pi}{2n},\ \frac{5\pi}{2n},\ \frac{7\pi}{2n},$$

quamdiu sunt recto minores, pro quibus ibi alternatim $+$ et $-$ scribi oportet; haeque partes omnes in unam summam collectae dabunt valorem completum ipsius y, dummodo pro ultima parte ex angulo $\zeta=\frac{\pi}{2}$ oriunda, quod evenit si n numerus impar, ejus tantum semissis capiatur.

Corollarium 1.

1212. Accommodemus haec statim ad casum $n=\infty$ et $a=nc$, ut proposita sit haec aequatio differentialis

$$X = y + \frac{\partial\partial y}{1.2\, c^2\, \partial x^2} + \frac{\partial^4 y}{1.2.3.4\, c^4\, \partial x^4} + \frac{\partial^6 y}{1\ldots 6\, c^6\, \partial x^6} + \text{etc. in infinitum.}$$

Cum igitur hic valores ipsius ζ sint infinite parvi, erit

$$\cos.\ \zeta = 1 \quad \text{et} \quad \text{tang.}\ \zeta = \zeta = \frac{(4i \pm 1)\pi}{2n},$$

hinc $a\,x$ tang. $= (4\,i \pm 1)\,c\,x \,.\, \frac{\pi}{2}$,

pro quo angulo scribamus ω. Ergo pars integralis quaecunque

$$\pm 2\,c\,(\sin.\,\omega \int X\,\partial x \cos.\,\omega - \cos.\,\omega \int X\,\partial x \sin.\,\omega),$$

ubi signa ambigua sibi mutuo respondent.

Corollarium 2.

1213. Si tantum angulus $\frac{\pi}{2}cx$ ponatur $= \Phi$, integrale universum ita erit expressum

$$\begin{aligned}\frac{y}{2c} = &+ \sin.\,\Phi \int X\,\partial x \cos.\,\Phi - \cos.\,\Phi \int X\,\partial x \sin.\,\Phi \\ &- \sin.\,3\,\Phi \int X\,\partial x \cos.\,3\,\Phi + \cos.\,3\,\Phi \int X\,\partial x \sin.\,3\,\Phi \\ &+ \sin.\,5\,\Phi \int X\,\partial x \cos.\,5\,\Phi - \cos.\,5\,\Phi \int X\,\partial x \sin.\,5\,\Phi \\ &- \sin.\,7\,\Phi \int X\,\partial x \cos.\,7\,\Phi + \cos.\,7\,\Phi \int X\,\partial x \sin.\,7\,\Phi \\ &\text{etc.}\end{aligned}$$

quae formulae in infinitum sunt continuandae.

Corollarium 3.

1214. Si ponamus $c = b\sqrt{-1}$, ut habetur haec aequatio infinita

$$X = y - \frac{\partial\partial y}{1.2\, b^2\, \partial x^2} + \frac{\partial^4 y}{1\ldots 4\, b^4\, \partial x^4} - \frac{\partial^6 y}{1\ldots 6\, b^6\, \partial x^6} + \text{etc.}$$

ac jam angulum $\frac{\pi}{2}\,b\,x$ vocemus ψ, erit integrale completum

$$\frac{y}{b} = + e^{-\psi} \int e^{\psi} X \partial x - e^{\psi} \int e^{-\psi} X \partial x$$
$$- e^{-3\psi} \int e^{3\psi} X \partial x + e^{3\psi} \int e^{-3\psi} X \partial x$$
$$+ e^{-5\psi} \int e^{5\psi} X \partial x - e^{5\psi} \int e^{-5\psi} X \partial x$$
etc.

Scholion.

1215. Si pro aequatione Corollarii 1. methodo supra exposita quaeramus integrale particulare per differentialia ipsius X expressum, huncque in finem ponamus

$$y = AX - \frac{B\partial\partial X}{c^2 \partial x^2} + \frac{C\partial^4 X}{c^4 \partial x^4} - \frac{D\partial^6 X}{c^6 \partial x^6} + \frac{E\partial^8 X}{c^8 \partial x^8} - \text{etc.}$$

reperiemus hos coëfficientium valores

$$A = 1,\ B = \frac{1}{1.2},\ C = \frac{5}{1\ldots4},\ D = \frac{61}{1\ldots\ldots6},$$
$$E = \frac{1385}{1\ldots\ldots8},\ F = \frac{50521}{1\ldots\ldots10},\ \text{etc.}$$

Hicque valor si ponatur $= V$, vocato angulo $\frac{\pi}{2} c x = \Phi$, erit integrale completum

$$y = V + A \sin.(\alpha + \Phi) + B \sin.(\beta + 3\Phi) + C \sin.(\gamma + 5\Phi)$$
$$+ D \sin.(\delta + 7\Phi) + \text{etc.}$$

Problema 162.

1216. Proposita aequatione differentiali

$$X = y + \frac{n(n-1)}{1.2a^2} \cdot \frac{\partial y}{\partial x} + \frac{n(n-1)(n-2)(n-3)}{1.2.3.4a^4} \cdot \frac{\partial\partial y}{\partial x^2}$$
$$+ \frac{n\ldots\ldots(n-5)}{1\ldots6a^6} \cdot \frac{\partial^3 y}{\partial x^3} + \text{etc.}$$

ejus integrale completum investigare.

Solutio.

Quantitas algebraica hinc formanda

$$P = 1 + \frac{n(n-1)}{1.2a^2}z + \frac{n(n-1)(n-2)(n-3)}{1.\ 2.\ 3.\ 4\ a^4}.zz + \text{etc.}$$
$$= \tfrac{1}{2}\left(1+\frac{\sqrt{z}}{a}\right)^n + \tfrac{1}{2}\left(1-\frac{\sqrt{z}}{a}\right)^n,$$

cum ex praecedente nascatur, si ibi loco zz scribatur z, sumto angulo $\zeta = \frac{2i+1}{2n}\pi$, factor quicunque erit $aa\,\text{tang.}\,\zeta^2 + z$, ita ut hujus formae omnes factores simplices sint reales. Hoc ergo factore cum formula $\alpha + z$ comparato, erit $\alpha = aa\,\text{tang.}\,\zeta^2$, et sumto $\mathfrak{A} = \frac{P}{\alpha+z}$ posito $z = -\alpha$, erit integralis pars ex hoc factore oriunda $\frac{1}{\mathfrak{A}}e^{-\alpha x}\int e^{\alpha x}X\,\partial x$. Quia vero P evanescit posito $z = -\alpha$, erit quoque $\mathfrak{A} = \frac{\partial P}{\partial z}$; at est differentiando

$$\frac{\partial P}{\partial z} = \frac{n}{4a\sqrt{z}}\left[\left(1+\frac{\sqrt{z}}{a}\right)^{n-1} - \left(1-\frac{\sqrt{z}}{a}\right)^{n-1}\right].$$

Quia igitur poni oportet $\frac{\sqrt{z}}{a} = \text{tang.}\,\zeta\,.\sqrt{-1}$, erit

$$1 + \frac{\sqrt{z}}{a} = \frac{\cos.\zeta + \sqrt{-1}.\sin.\zeta}{\cos.\zeta} \text{ et } 1 - \frac{\sqrt{z}}{a} = \frac{\cos.\zeta - \sqrt{-1}.\sin.\zeta}{\cos.\zeta}$$

hincque

$$\mathfrak{A} = \frac{n}{4aa\,\text{tang.}\,\zeta.\sqrt{-1}} \cdot \frac{2\sqrt{-1}.\sin.(n-1)\zeta}{\cos\,\zeta^{n-1}} = \frac{n\sin.(n-1)\zeta}{2aa\sin.\zeta\cos.\zeta^{n-2}}.$$

Jam observetur esse $\sin.\,n\zeta = \sin.\,(2i+1)\frac{\pi}{2} = \pm 1$, (ubi signum superius valet si i numerus par, inferius si impar,) tum vero $\cos.\,n\zeta = 0$, unde fit $\sin.\,(n-1)\zeta = \pm\cos.\,\zeta$, ex quo conficitur

$$\mathfrak{A} = \frac{\pm n}{2aa\sin.\zeta\cos.\zeta^{n-3}},$$

et pars integralis quaesita habebitur

$$\pm\frac{2aa\sin.\zeta\cos.\zeta^{n-3}}{n}e^{-aa\,\text{tang.}\,\zeta^2.x}\int e^{aa\,\text{tang.}\,\zeta^2.x}X\,\partial x.$$

Nunc igitur ipsi ζ successive tribuantur hi valores

$$\frac{\pi}{2n},\ \frac{3\pi}{2n},\ \frac{5\pi}{2n},\ \frac{7\pi}{2n},\ \text{etc.}$$

quoad angulum rectum non superent, atque omnes istae partes

in unam summam collectae, dabunt integrale completum seu valorem ipsius y.

Corollarium 1.

1217. Si ponamus $n = \infty$ et $a = nc$, aequatio proposita in infinitum excurrit, eritque

$$X = y + \frac{\partial y}{1.2c^2 \partial x} + \frac{\partial\partial y}{1.2.3.4c^4\partial x^2} + \frac{\partial^3 y}{1\ldots 6c^6\partial x^3} + \text{etc.}$$

et forma algebraica inde nata

$$P = 1 + \frac{z}{1.2c^2} + \frac{zz}{1.2.3.4c^4} + \frac{z^3}{1\ldots 6c^6} + \text{etc.} = \tfrac{1}{2}e^{\frac{\sqrt{z}}{c}} + \tfrac{1}{2}e^{\frac{-\sqrt{z}}{c}},$$

quae omnes factores simplices habet reales, et ob ζ infinite parvum erit tang. $\zeta = \zeta = \frac{2i+1}{2n}\pi$, indeque factorum forma generalis

$$z + \frac{(2i+1)^2}{4}\pi\pi cc, \text{ seu } 1 + \frac{4z}{(2i+1)^2\pi\pi cc}.$$

Corollarium 2.

1218. Ponatur brevitatis gratia angulus

$\frac{2i+1}{2}\pi = \theta$, erit

aa tang $\zeta^2 = \theta\theta cc$, tum vero

cos. $\zeta = 1$ et $\frac{aa\sin.\zeta}{n} = \theta cc$,

ex quo integralis pars quaecunque erit

$$\pm 2\theta cce^{-\theta\theta ccx}\int e^{\theta\theta ccx}X\partial x,$$

ubi pro θ successive omnes hos angulos scribi oportet

$\frac{\pi}{2}$, $\frac{3\pi}{2}$, $\frac{5\pi}{2}$, $\frac{7\pi}{2}$, $\frac{9\pi}{2}$, etc.

Corollarium 3.

1219. Perinde hic est sive cc negative sive positive capiatur, hinc istius aequationis differentialis infinitae

$$X = y + \frac{\partial y}{1.2b\partial x} + \frac{\partial\partial y}{1.2.3.4bb\partial x^2} + \frac{\partial^3 y}{1\ldots 6b^3\partial x^3} + \text{etc.}$$

integrale erit

$$y = 2\theta b e^{-\theta\theta bx} \int e^{\theta\theta bx} X \partial x,$$

loco θ scribendo successive omnes hos angulos, ambiguitate signi jam sublata

$$+\frac{\pi}{2},\ -\frac{3\pi}{2},\ +\frac{5\pi}{2},\ -\frac{7\pi}{2},\ + \text{etc.}$$

unde si $X = 0$, integrale particulare quodvis est

$$y = A e^{-\theta\theta bx}.$$

Problema 163.

1220. Proposita aequatione differentiali

$$X = \frac{n\partial y}{a\partial x} + \frac{n(n-1)(n-2)\partial\partial y}{1.2.3.a^3\partial x^2} + \frac{n(n-1)(n-2)(n-3)(n-4)\partial^3 y}{1.\ 2.\ 3.\ 4.\ 6 a^6 \partial x^3} + \text{etc.}$$

ejus integrale completum investigare.

Solutio.

Etsi haec aequatio in ∂x ducta sponte semel integratur, praestat tamen hanc formam retinere, unde fit

$$P = \frac{nz}{a} + \frac{n(n-\cdot)(n-2)zz}{1.\ 2.\ 3\ a^3} + \frac{n(n-1)(n-2)(n-3)(n-4)z^3}{1.\ 2.\ 3.\ 4.\ 5\ a^5} + \text{etc.}$$

quae manifesto ita exhiberi potest

$$P = \frac{\sqrt{z}}{2}\left[\left(1 + \frac{\sqrt{z}}{a}\right)^n - \left(1 - \frac{\sqrt{z}}{a}\right)^n\right],$$

cujus quidem statim unus factor se offert z; reliqui vero in hac forma continentur

$$\left(1 + \frac{\sqrt{z}}{a}\right)^2 - 2\left(1 - \frac{z}{aa}\right)\cos. 2\zeta + \left(1 - \frac{\sqrt{z}}{a}\right)^2,$$

sumto angulo

$$2\zeta = \frac{2i\pi}{n} \quad \text{seu} \quad \zeta = \frac{i\pi}{n},$$

haec vero forma abit in

$$2(1 - \cos. 2\zeta) + \frac{2z}{aa}(1 + \cos. 2\zeta),$$

unde patet in genere factorem fore $a a$ tang. $\zeta^2 + z$, quae etiam primum illum z complectitur sumto $i = 0$. Hinc posito $a a$ tang. $\zeta^2 = \alpha$, integralis pars huic factori respondens erit

$$\tfrac{1}{\mathfrak{A}} e^{-\alpha x} \int e^{\alpha x} X \partial x,$$

si posito

$$z = - a a \text{ tang. } \zeta^2 \text{ seu } \sqrt{z} = a \text{ tang. } \zeta \sqrt{-1},$$

capiatur

$$\mathfrak{A} = \frac{\partial P}{\partial z} = \frac{1}{4\sqrt{z}} \left[\left(1 + \frac{\sqrt{z}}{a}\right)^n - \left(1 - \frac{\sqrt{z}}{a}\right)^n \right]$$
$$+ \frac{n}{4a} \left[\left(1 + \frac{\sqrt{z}}{a}\right)^{n-1} + \left(1 - \frac{\sqrt{z}}{a}\right)^{n-1} \right]. \text{ At}$$

$$\left(1 + \frac{\sqrt{z}}{n}\right)^n = \frac{\cos. n\zeta + \sqrt{-1} . \sin. n\zeta}{\cos. \zeta^n}, \text{ et } \left(1 - \frac{\sqrt{z}}{a}\right)^n = \frac{\cos. n\zeta - \sqrt{-1} . \sin. n\zeta}{\cos \zeta^n},$$

quamobrem fiet

$$\mathfrak{A} = - \frac{\sin. n\zeta}{2 a \text{ tang } \zeta \cos. \zeta^n} + \frac{n \cos. (n-1) \zeta}{2 a \cos. \zeta^{n-1}} = \frac{+ n}{2 a \cos. \zeta^{u-2}},$$

ob $\sin. n\zeta = 0$ et $\cos. n\zeta = \pm 1$, prout numerus i fuerit vel par vel impar. Quocirca integralis pars quaecunque ita erit expressa

$$\pm \frac{2 a \cos. \zeta^{n-2}}{n} e^{-\alpha x} \int e^{\alpha x} X \partial x,$$

existente $\alpha = a a$ tang. ζ^2. Jam angulo ζ successive tribuantur hi valores

$$\frac{0\pi}{n}, \frac{1\pi}{n}, \frac{2\pi}{u}, \frac{3\pi}{n},$$

quoad angulum rectum $\frac{\pi}{2}$ non excedant, haeque formulae omnes cum suis signis in unam summam conjectae dabunt valorem completum pro y.

Corollarium 1.

1221. Prima igitur integralis pars nascitur ex angulo $\zeta = 0$, unde ea erit $+ \frac{2a}{n} \int X \partial x$, cujus autem loco ob rationes supra

allegatas circa factores simplices, ejus tantum dimidium sumi debet, ut haec prima pars sit $= \frac{a}{n} \int X \partial x$, quod etiam inde patet, quod posito $z = 0$ fiat manifesto $\frac{P}{a} = \frac{n}{a}$.

Corollarium 2.

1222. Idem tenendum esset de parte ultima, siquidem ex valore $\zeta = \frac{\pi}{2}$ nascatur, quod evenit si n sit numerus par. Quia vero hoc casu fit $\cos. \zeta = 0$, haec tota integralis pars per se evanescit.

Corollarium 3.

1223. Si esset $X = 0$, quaelibet pars integralis foret $A e^{-aa \,tang. \zeta^2 x}$, denotante A quantitatem constantem arbitrariam; foretque adeo haec aequatio

$$y = A e^{-aa \,tang. \zeta^2 x}$$

integrale particulare aequationis, dummodo capiatur angulus

$$\zeta = \frac{i\pi}{n}.$$

Scholion.

1224. Hinc posito $n = \infty$ et $a = n\sqrt{b}$, integrari potest haec aequatio differentialis in infinitum excurrens

$$\frac{X}{\sqrt{b}} = \frac{\partial y}{1 \cdot b \partial x} + \frac{\partial\partial y}{1 \cdot 2 \cdot 3 bb \partial x^2} + \frac{\partial^3 y}{1 \ldots\ldots 5 b^2 \partial x^3} + \frac{\partial^4 y}{1 \ldots\ldots 7 b^4 \partial x^4} + \text{etc.}$$

vel etiam haec per unam integrationem ex ista nata

$$\sqrt{b} \,.\int X \partial x = \frac{y}{1} + \frac{\partial y}{1 \cdot 2 \cdot 3 b \partial x} + \frac{\partial\partial y}{1 \ldots 5 b^2 \partial x^2} + \frac{\partial^3 y}{1 \ldots\ldots 7 b^3 \partial x^3} + \text{etc.}$$

Cum enim sit angulus $\zeta = \frac{i\pi}{n}$ infite parvus, erit

$$\cos. \zeta = 1, \text{ et } a \text{ tang. } \zeta = a\zeta = i\pi\sqrt{b},$$

ideoque

$$\alpha = aa \text{ tang. } \zeta^2 = ii\pi\pi b,$$

habebitur pars integralis quaecunque

$$\pm 2\sqrt{b}\,.\,e^{-ii\pi\pi bx}\int e^{ii\pi\pi bx}X\partial x,$$

unde parte prima ex $i=0$ nata ad dimidium reducta, ab rationes supra allegatas, erit integrale completum

$$\frac{y}{\sqrt{b}}=\int X\partial x-2e^{-\pi\pi bx}\int e^{\pi\pi bx}X\partial x+2e^{-4\pi\pi bx}\int e^{4\pi\pi bx}X\partial x$$
$$-2e^{-9\pi\pi bx}\int e^{9\pi\pi bx}X\partial x+2e^{-16\pi\pi bx}\int e^{16\pi\pi bx}X\partial x-\text{etc.}$$

Exemplum.

1225. Sit $n=6$ et $a=1$, ut integranda proponatur haec aequatio

$$X=\frac{6\partial y}{\partial x}+\frac{20\partial\partial y}{\partial x^2}+\frac{6\partial^3 y}{\partial x^3},\ \text{seu}$$
$$\int X\partial x=6y+\frac{20\partial y}{\partial x}+\frac{6\partial\partial y}{\partial x^2}.$$

Valores ergo pro angulo ζ et inde pendentes sunt

$$\zeta=0,\ 30^\circ,\ 60^\circ$$
$$\cos.\zeta=1,\ \frac{\sqrt{3}}{2},\ \frac{1}{2},$$
$$\alpha=0,\ \frac{1}{3},\ 3,$$

ex quibus colligitur integrale quaesitum

$$y=\frac{1}{6}\int X\partial x-\frac{3}{16}e^{-\frac{1}{3}x}\int e^{\frac{1}{3}x}X\partial x+\frac{1}{48}e^{-3x}\int e^{3x}X\partial x,$$

quod etiam aequationi satisfacere tentanti patebit.

CAPUT V.

DE

INTEGRATIONE AEQUATIONUM DIFFERENTIALIUM HUJUS FORMAE

$$X = Ay + \frac{Bx\partial y}{\partial x} + \frac{Cx^2\partial\partial y}{\partial x^2} + \frac{Dx^3\partial^3 y}{\partial x^3} + \frac{Ex^4\partial^4 y}{\partial x^4} + \text{etc.}$$

Problema 164.

1226.

Proposita aequatione differentiali hujus formae

$$X = Ay + \frac{Bx\partial y}{\partial x} + \frac{Cxx\partial\partial y}{\partial x^2} + \frac{Dx^3\partial^3 y}{\partial x^3} + \ldots + \frac{Nx^n\partial^n y}{\partial x^n},$$

definire functionem ipsius x, per quam ea multiplicata fiat integrabilis.

Solutio.

Attendenti mox patebit, simdlicem potestatem ipsius x hoc praestare. Sit igitur integrabilis haec aequatio

$$Xx^\lambda\partial x = Ax^\lambda y\partial x + Bx^{\lambda+1}\partial y + \frac{Cx^{\lambda+2}\partial\partial y}{\partial x^2} + \ldots + \frac{Nx^{\lambda+n}\partial^n y}{\partial x^{n-1}},$$

cujus integrale sit

$$\int Xx^\lambda\partial x = A'x^{\lambda+1}y + \frac{B'x^{\lambda+2}\partial y}{\partial x} + \frac{C'x^{\lambda+3}\partial\partial y}{\partial x^2} + \ldots + \frac{M'x^{\lambda+n}\partial^{n-1}y}{\partial x^{n-1}}.$$

Cum igitur hujus differentiale illi debeat esse aequale, sequentes nanciscemur determinationes

$A = (\lambda+1)A'$, hinc $(\lambda+1)A' = A$

$B = (\lambda+2)B' + A'$, $(\lambda+1)(\lambda+2)B' = (\lambda+1)B - A$

$C = (\lambda+3)C' + B'$, $(\lambda+1)(\lambda+2)(\lambda+3)C' = (\lambda+1)(\lambda+2)C - (\lambda+1)B + A$

$$D = (\lambda+4)D' + C', (\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)D' = (\lambda+1)(\lambda+2)(\lambda+3)D - (\lambda+1)(\lambda+2)C + (\lambda+1)B - A$$

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$N = M'$

integralis enim termini sequentes, qui involuerent differentialis gradum $\partial^n y$ altioresque, evanescere debent, quia alioquin integratio non successisset. Cum igitur in integrali littera N' evanescat, pervenimus ad hanc aequationem

$$0 = A - (\lambda+1)B + (\lambda+1)(\lambda+2)C - (\lambda+1)(\lambda+2)(\lambda+3)D \ldots\ldots \pm (\lambda+1)(\lambda+2)\ldots\ldots(\lambda+n)N$$

ex qua aequatione exponens λ potestatis quaesitae x^λ definiri debet. Formetur ergo talis expressio algebraica

$$P = A + B(z-1) + C(z-1)(z-2) + D(z-1)(z-2)(z-3) + \ldots\ldots + N(z-1)(z-2)(z-3)\ldots\ldots(z-n),$$

hujusque quaerantur omnes factores simplices, ut sit

$$P = N(\alpha+z)(\beta+z)(\gamma+z)(\delta+z) \text{ etc.}$$

factorum horum numero existente $= n$. Jam ex quolibet factore $z+\alpha$ ad nihilum reducto, valor $z = -\alpha$ dabit potestatem x^α, per quam proposita aequatio multiplicata integrabilis evadit, ita ut ejus integrale sit futurum

$$x^{-\alpha-1}\int x^\alpha X \partial x = A'y + \frac{B'x\partial y}{\partial x} + \frac{C'x^2\partial\partial y}{\partial x^2} + \frac{D'x^3\partial^3 y}{\partial x^3} + \ldots + \frac{Nx^{n-1}\partial^{n-1}y}{\partial x^{n-1}}$$

ubi differentialium gradus unitate est inferior. Ita autem haec aequatio integrata per propositam determinatur, ut sit

$$A = (\alpha + 1) A'$$
$$B = (\alpha + 2) B' + A'$$
$$C = (\alpha + 3) C' + B'$$
$$D = (\alpha + 4) D' + C'$$
etc.

donec perveniatur ad ultimum coëfficientem N, qui utrobique est idem.

Corollarium 1.

1227. Quia aequatio integrata similis est ipsi propositae. ea per certam potestatem ipsius x multiplicata denuo fiet integrabilis. Ad hanc enim potestatem inveniendam considerare oportet hanc formam algebraicam

$$Q = A' + B'(z-1) + C'(z-1)(z-2) + D'(z-1)(z-2)(z-3) + \ldots$$
$$\ldots\ldots + N(z-1)(z-2)\ldots\ldots(z-n+1),$$

cujus si fuerit factor simplex quicunque $z + \mu$, erit $x^{+\mu}$ illa potestas ipsius x, aequationem integrabilem reddens.

Corollarium 2.

1228. Quodsi ipsa aequatio proposita per potestatem $x^{+\alpha}$ multiplicata reddita fuerit integrabilis, hic probe notari convenit, quantitatem Q ex integrata formatam ita pendere a priori P ex ipsa proposita formata, ut sit $Q = \frac{P}{\alpha + z}$, quandoquidem per hypothesin $\alpha + z$ est factor ipsius P.

Scholion 1.

1229. Ad hanc insignem proprietatem demonstrandam, quod scilicet sit $P = (\alpha + z) Q$, tantum opus est, ut quantitas Q per

$\alpha + z$ multiplicetur; verum quo conclusio clarius in oculos occurrat, pro singulis terminis ipsius Q multiplicator bipartito est repraesentandus; ac pro primo quidem termino loco $\alpha + z$ scribatur $(\alpha + 1) + (z - 1)$, pro secundo $(\alpha + 2) + (z - 2)$, pro tertio $(\alpha + 3) + (z - 3)$, pro quarto $(\alpha + 4) + (z - 4)$ etc. ita ut cujusque termini productum binis partibus exhibeatur, quam operationem hic totam apponam

$$Q = A' + B'(z-1) + C'(z-1)(z-2) + D'(z-1)(z-2)(z-3) + \text{etc.}$$

Multipl.	$\alpha + 1$	$\alpha + 2$	$\alpha + 3$	$\alpha + 4$
	$z - 1$	$z - 2$	$z - 3$	$z - 4$

Prod. $(\alpha+1)A' + A'(z-1) \quad + B'(z-1)(z-2) \quad + C'(z-1)(z-2)(z-3)$ etc.

$+ (\alpha+2)B'(z-1) + (\alpha+3)C'(z-1)(z-2) + (\alpha+4)D'(z-1)(z-2)(z-3)$

In solutione autem vidimus esse

$$(\alpha+1)A' = A, \quad (\alpha+2)B' + A' = B, \quad (\alpha+3)C' + B' = C, \text{ etc.}$$

quocirca hoc productum hac forma exprimetur

$$A + B(z-1) + C(z-1)(z-2) + D(z-1)(z-2)(z-3) + \text{etc.}$$

cui valor ipsius P est aequalis, sicque demonstrata est insignis illa proprietas memorata, quod sit $Q = \frac{P}{\alpha + z}$.

Corollarium 3.

1230. Quodsi ergo valor ipsius P in factores simplices resolutus ita repraesentetur

$$P = N(\alpha + z)(\beta + z)(\gamma + z)(\delta + z) \text{ etc.}$$

et ex factore $\alpha + z$ aequatio proposita per x^2 multiplicata integretur, tum vero ex integratata simili modo valor Q formetur, erit

$$Q = N(\beta + z)(\gamma + z)(\delta + z) \text{ etc.}$$

Corollarium 4.

1231. Aequatio ergo integrata, postquam ad formam propositae fuerit perducta, ut posito

$$x^{-\epsilon-1}\int x^{\epsilon} X \partial x = X'$$

habeatur

$$X' = A' y + \frac{B' x \partial x}{\partial x} + \frac{C' x^2 \partial \partial y}{\partial x^2} + \frac{D' x^3 \partial^3 y}{\partial x^3} + \text{etc.}$$

haec denuo integrabilis reddetur, si multiplicetur per quampiam harum potestatum x^{β}, x^{γ}, x^{δ}, etc. quae etiam ipsam propositam integrabilem reddidissent.

Scholion 2.

1232. Antequam continuationem harum integrationum ulterius prosequar, conveniet eum casum formae propositae generalis seorsim evolvi, quo prius aequationis membrum X in nihilum abit. Hoc enim casu hoc commodi usu venit, ut statim sine integrationibus repetitis integrale completum exhiberi queat, idque simili modo quo supra in Capite II. sum usus. Huic quidem casui quia jam multo facilius tractari potest, proprium caput assignare nolui, ne praecepta nimis multiplicari videantur.

Problema 165.

1233. Proposita hac aequatione differentiali cujuscunque ordinis

$$0 = Ay + \frac{B x \partial y}{\partial x} + \frac{C x^2 \partial \partial y}{\partial x^2} + \frac{D x^3 \partial^3 y}{\partial x^3} + \text{etc.}$$

ubi variabilis y cum suis differentialibus nusquam plus una dimensione, altera vero x adeo nullam obtineat, ejus integrale completum invenire.

Solutio.

Particulariter huic aequationi satisfieri perspicuum est, si y certae ipsius x potestati aequetur, ponamus ergo esse $y = x^{\mu}$, et

facta substitutione, cum ubique per x^{μ} diviserimus, perveniemus ad hanc aequationem

$$0 = A + \mu B + \mu(\mu - 1)C + \mu(\mu - 1)(\mu + 2)D + \text{etc.}$$

unde exponentem μ determinari oportet. Vel si secundum praeceptum praecedentis Problematis ex aequatione proposita hanc formemus formulam algebraicam

$$P = A + B(z-1) + C(z-1)(z-2) + D(z-1)(z-2)(z-3) + \text{etc.}$$

eamque in factores simplices resolvamus, ut sit

$$P = N(\alpha + z)(\beta + z)(\gamma + z)(\delta + z) \text{ etc.}$$

evidens est posito $\mu = z - 1$, primae aequationi satisfieri sumendo

$$\mu = -\alpha - 1, \text{ vel } \mu = -\beta - 1, \text{ vel } \mu = -\gamma - 1, \text{ etc.}$$

ita ut quisque factor suppeditet integrale particulare. Cum igitur factorum numerus aequetur gradui differentialium summo, hinc colligetur integrale completum aequationis propositae

$$y = \mathfrak{A} x^{-\alpha-1} + \mathfrak{B} x^{-\beta-1} + \mathfrak{C} x^{-\gamma-1} + \mathfrak{D} x^{-\delta-1} + \text{etc.}$$

ubi tantum observari convenit, si factorum illorum simplicium duo pluresve fuerint inter se aequales, integralis formam simili modo immutari debere, quo supra Capite II. sum usus. Scilicet cum aequationes ibi tractatae ad praesentem formam revocentur, si ibi loco x scribatur lx, inde has regulas haurimus.

1) Si forma P factorem habeat $(\alpha + z)^2$, inde nascitur pars integralis

$$x^{-\alpha-1}(\mathfrak{A} + \mathfrak{B}\, lx).$$

2) Si forma P factorem habeat $(\alpha + z)^3$, pars integralis inde orta est

$$x^{-\alpha-1}[\mathfrak{A} + \mathfrak{B}\, lx + \mathfrak{C}(lx)^2].$$

3) Si forma P factorem habeat $(\alpha + z)^4$, pars integralis inde orta erit

$$x^{-\alpha-1}[\mathfrak{A} + \mathfrak{B}\, lx + \mathfrak{C}(lx)^2 + \mathfrak{D}(lx)^3].$$

Si factores occurrant imaginarii, partes inde oriundae per solitam imaginariorum reductionem facile ad formam realem revocabuntur, uti in corollariis docebo.

Corollarium 1.

1234. Si forma P duos habeat factores simplices imaginarios in formula $ff + 2fz\cos.\theta + zz$ contentos, hac cum producto $(\alpha + z)(\beta + z)$ comparata erit

$$\alpha = f(\cos.\theta + \sqrt{-1}.\sin.\theta) \text{ et } \beta = f(\cos.\theta - \sqrt{-1}.\sin.\theta),$$

unde fit

$$x^{-\alpha} = x^{-f\cos.\theta}\, x^{-f\sqrt{-1}.\sin\theta} = x^{-f\cos.\theta}\, e^{-\sqrt{-1}.f\sin.\theta\, lx}.$$

Est vero

$$e^{-u\sqrt{-1}} = \cos.u - \sqrt{-1}.\sin.u,$$

ideoque habetur

$$x^{-\alpha-1} = x^{-f\cos.\theta}\left(\frac{\cos.(f\sin.\theta\, lx) - \sqrt{-1}.\sin.(f\sin.\theta\, lx)}{x}\right).$$

Quare cum $x^{-\beta-1}$ simili modo exprimatur, mutato signo ipsius $\sqrt{-1}$, ex factore duplici $ff + 2fz\cos.\theta + zz$ haec nascitur pars integralis

$$x^{-f\cos.\theta-1}[\mathfrak{A}\cos.(f\sin.\theta\, lx) + \mathfrak{B}\sin.(f\sin.\theta\, lx)];$$

quae etiam ita potest repraesentari

$$\mathfrak{A}\, x^{-f\cos.\theta-1}\cos.(\mathfrak{a} + f\sin.\theta\, lx),$$

denotante $\mathfrak{a}$ angulum constantem arbitrarium.

Corollarium 2.

1235. Simili modo si factores aequales involuat, ut sit

$$(\alpha + z)^2(\beta + z)^2 = (ff + 2fz\cos.\theta + zz)^2,$$

litterae α et β eosdem quos ante sortientur valores imaginarios, ex quorum reductione colligitur haec pars integralis inde oriunda.

$$x^{-f\cos.\theta - a}\left[\mathfrak{A}\cos.(\mathfrak{a} + f\sin.\theta\, lx) + \mathfrak{B}\, lx\,.\cos.(\mathfrak{b} + f\sin.\theta\, lx)\right],$$

quatuor constantes arbitrarias $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{a}$ et $\mathfrak{b}$ continens.

Corollarium 3.

1136. Hinc ergo evidens est, quomodo ex factoribus formae P; sive sint simplices sive duplices, sive inaequales sive aequales, singulas integralis partes assignari indeque totum integrale completum formari conveniat.

Scholion.

1237. Totum ergo negotium huc redit, ut quantitas algebraica ex aequatione differentiali formata

$$P = A + B(z-1) + C(z-1)(z-2) + D(z-1)(z-2)(z-3) + \text{etc.}$$

in suos factores reales vel simplices vel duplices resolvetur, in quo plerumque maxima difficultas versatur, quoniam hujusmodi formae minus tractari sunt solitae. Cum vero haec resolutio isti aequationi cum generali, quam hoc capite evolvere suscepi, est communis, quicquid hic praestare licuerit, potius in aequatione generali ostendi conveniet, ad quam resolvendam propterea revertor. Id tantum hic observasse necesse duco, quod si pro aequatione generali

$$X = Ay + \frac{Bx\partial y}{\partial x} + \frac{Cxx\partial\partial y}{\partial x^2} + \frac{Dx^3\partial^3 y}{\partial x^3} + \text{etc.}$$

undecunque innotuerit integrale particulare, puta $y = V$ existente V certa functione ipsius x, tum posito $y = V + v$ perveniri ad hanc aequationem

$$0 = Av + \frac{Bx\partial v}{\partial x} + \frac{Cxx\partial\partial v}{\partial x^2} + \frac{Dx^3\partial^3 v}{\partial x^3} + \text{etc.}$$

cujus integrale completum per praecepta hujus problematis inventum si loco v scribatur, habebitur integrale completum illius aequationis, quo pacto certe insigne calculi compendium obtinetur.

Problema 166.

1238. **Proposita aequatione differentiali gradus cujuscunque n hujus formae**

$$X = Ay + \frac{Bx\partial y}{\partial x} + \frac{Cx^2\partial\partial y}{\partial x^2} + \ldots + \frac{Nx^n\partial^n y}{\partial x^n},$$

ejus integrale per integrationem n vicibus repetitam invenire.

Solutio.

Ex hac aequatione formetur haec quantitas algebraica
$P = A + B(z-1) + C(z-1)(z-2) + \ldots + N(z-1)(z-2)\ldots(z-n)$,
cujus quaerantur omnes factores simplices, nullo habito respectu sive sint reales sive imaginarii, ut ea hoc modo exprimatur

$$P = N(\alpha+z)(\beta+z)(\gamma+z)\ldots(\mu+z)(\nu+z),$$

factorum numero existente $= n$. Quo facto, initio hujus capitis vidimus, quemlibet factorem puta $\alpha + z$ praebere potestatem x^α, per quam nostra aequatio fiat integrabilis, atque adeo ostendimus, integrale inde ortum, si compendii gratia ponamus

$$x^{-\alpha-1}\int x^\alpha X\partial x = X',$$

fore

$$X' = A'y + \frac{B'x\partial y}{\partial x} + \frac{C'x^2\partial\partial y}{\partial x^2} + \ldots + \frac{Nx^{n-1}\partial^{n-1}y}{\partial x^{n-1}},$$

ita ut sit $A' = \frac{A}{\alpha+1}$, caeterique coëfficientes ita se habeant, uti ibidem docuimus; hic autem sufficiet ad primum potissimum respexisse. Absoluta jam prima integratione, si eadem lege ex aequatione semel integrata formemus quantitatem
$P' = A' + B'(z-1) + C'(z-1)(z-2) + \ldots + N(z-1)(z-2)\ldots(z-n+1)$,
cujus resolutio in factores jam ex prima forma P constat, postquam §. 1229 demonstravi esse

$$P' = N(\beta + z)(\gamma + z)(\delta + z)\ldots(\mu + z)(\nu + z),$$

ita ut sit $P' = \frac{P}{\alpha + z}$. Hinc ergo simili modo factor $\beta + z$ suppeditabit multiplicatorem x^β, quo haec aequatio integrabilis redditur, ac posito

$$x^{-\beta-1} \int x^\beta X' \partial x = X'',$$

ut sit

$$X'' = x^{-\beta-1} \int e^{\beta-\alpha-1} \partial x \int x^\alpha X \partial x,$$

integrale erit

$$X'' = A'' y + \frac{B'' x \partial y}{\partial x} + \frac{C'' x^2 \partial \partial y}{\partial x^2} + \ldots + \frac{N x^{n-2} \partial^{n-2} y}{\partial x^{n-2}},$$

existente hic $A'' = \frac{A'}{\beta + 1} = \frac{A}{(\alpha + 1)(\beta + 1)}$. Quodsi hoc modo tot integrationes successive absolvantur, quot unitates continentur in indice n, sicque omnes factores simplices formae P in usum vocentur, tandem ad hanc pervenietur aequationem $X^{(n)} = A^{(n)} y$, quae est ipsa integralis desiderata. Cum autem hic futurum sit

$$A^{(n)} = \frac{A}{(\alpha + 1)(\beta + 1)(\gamma + 1)\ldots(\nu + 1)},$$

evidens est denominatorem nasci ex forma $\frac{P}{N}$, si loco z scribatur unitas; tum autem sumto $z = 1$, prima forma manifesto dat $P = A$; ita ut denominator iste fiat $= \frac{A}{N}$, ideoque $A^{(n)} = N$, quod etiam inde patet, quod omnium aequationum ultimi termini habeant eundem coëfficientem N, quo ergo in postrema integrali ipse primus terminus y debet esse affectus. Deinde vero est

$$X^{(n)} = x^{-\nu-1} \int x^{\nu-\mu-1} \partial x \int x^{\mu-\lambda-1} \partial x \ldots \int x^{\beta-\alpha-1} \partial x \int x^\alpha X \partial x,$$

ubi cum numeri α, β, γ, etc. utcunque inter se permutari possunt, integrale quaesitum etiam hoc modo repraesentari potest:

$$N y = x^{-\alpha-1} \int x^{\alpha-\beta-1} \partial x \int x^{\beta-\gamma-1} \partial x \ldots \int x^{\mu-\nu-1} \partial x \int x^\nu X \partial x$$

Corollarium 1.

1239. Totum ergo negotium huc redit, ut forma algebraica
$P = A + B(z-1) + C(z-1)(z-2) + D(z-1)(z-2)(z-3) + \text{etc.}$
in suos factores simplices resolvatur, quibus inventis ut sit

$$P = N(\alpha + z)(\beta + z)(\gamma + z) \ldots \ldots (\nu + z),$$

hinc integrale quaesitum facile exhibetur, et quidem pro factorum varia permutatione pluribus modis, qui autem omnes eundem valorem exprimunt, uti ex sequentibus clarius patebit.

Corollarium 2.

1240. Cum haec forma integralis inventa tot involuat integrationes, quoti gradus fuerit aequatio differentialis proposita, totidem quoque constantes arbitrariae ingerentur, quemadmodum indoles integrationis completae postulat.

Scholion.

1241. Quoniam integrale inventum pluribus integrationibus est involutum, ad usum faciliorem conveniet hanc formam in partes resolvi, quae singulae unicum tantum signum integrale contineant. Hanc autem resolutionem simili modo instituere licet, quo supra sumus usi, atque hic quidem totum negotium ad hujusmodi formulam revocatur

$$\int x^{m-n-1} \partial x \int x^n X \partial x,$$

quae manifesto ita reducitur, ut sit

$$\frac{1}{m-n} x^{m-n} \int x^n X \partial x - \frac{1}{m-n} \int x^m X \partial x:$$

ubi tamen observandum est, si fuerit $m = n$, peculiari reductione opus esse, hocque casu fore

$$\int \frac{\partial x}{x} \int x^n X \partial x = l x . \int x^n X \partial x - \int x^n X \partial x \, l x.$$

Hac ergo regula utemur in resolutione sequentium problematum, quibus successive omnes gradus differentialium percurramus, praetermisso quidem gradu primo, cum aequationis

$$X = Ay + \frac{Nx\partial y}{\partial x}, \text{ ob}$$

$$P = A + N(z - 1) = N(\alpha + z),$$

integrale sit

$$Ny = x^{-\alpha-1}\int x^{\alpha} X\partial x,$$

quod nulla reductione indiget.

Problema 167.

1242. Proposita hac aequatione differentiali secundi gradus

$$X = Ay + \frac{Bx\partial y}{\partial x} + \frac{Nx^2\partial\partial y}{\partial x^2},$$

ejus integrale per formulas integrales simplices evoluere.

Solutio.

Cum sit

$$P = A + B(z - 1) + N(z - 1)(z - 2),$$

statuatur

$$P = N(\alpha + z)(\beta + z),$$

eritque integrale per methodum praecedentem inventum $Ny = X''$ existente

$$x^{\beta+1}X'' = \int x^{\beta-\alpha-1}\partial x\int x^{\alpha} X\partial x,$$

quae forma evoluitur in hanc

$$\frac{1}{\beta-\alpha}x^{\beta-\alpha}\int x^{\alpha} X\partial x - \frac{1}{\beta-\alpha}\int x^{\beta} X\partial x,$$

sicque erit

$$Ny = \frac{x^{-\alpha-1}}{\beta-\alpha}\int x^{\alpha} X\partial x + \frac{x^{-\beta-1}}{\alpha-\beta}\int x^{\beta} X\partial x.$$

Hinc autem casum excipi oportet, quo $\beta = \alpha$, tum enim fit

$$x^{\alpha+1} X'' = \int \frac{\partial x}{x} \int x^\alpha X \partial x = lx \int x^\alpha X \partial x - \int x^\alpha X \partial x \, lx$$

pro hoc igitur casu habebimus

$$N y = x^{-\alpha-1} \int \frac{\partial x}{x} \int x^\alpha X \partial x, \text{ seu}$$

$$N y = x^{-\alpha-1} (lx \int x^\alpha X \partial x - \int x^\alpha X \partial x \, lx),$$

ubi quidem prior forma praeferenda videtur.

Corollarium 1.

1243. Si ambo factores simplices sint imaginarii, ponatur

$$(\alpha + z)(\beta + z) = ff + 2fz \cos.\theta + zz,$$

eritque

$$\alpha = f(\cos.\theta + \sqrt{-1}.\sin.\theta) \text{ et}$$
$$\beta = f(\cos.\theta - \sqrt{-1}.\sin.\theta),$$

indeque

$$\beta - \alpha = -2f\sqrt{-1}.\sin.\theta.$$

Tum vero

$$x^\alpha = x^{f\cos.\theta}[\cos.(f\sin.\theta.lx) + \sqrt{-1}.\sin.(f\sin.\theta.lx)]$$
$$x^{-\alpha} = x^{-f\cos.\theta}[\cos.(f\sin.\theta\, lx) - \sqrt{-1}.\sin.(f\sin.\theta\, lx)],$$

quae formulae mutato signo ipsius $\sqrt{-1}$ ad x^β et $x^{-\beta}$ transferuntur.

Corollarium 2.

1244. Ponatur brevitatis gratia angulus

$$f\sin.\theta\, lx = \Phi,$$

et facta substitutione habebimus

$$Nxy = \frac{x^{-f\cos.\theta}(\cos.\Phi - \sqrt{-1}.\sin.\Phi)\int x^{f\cos.\theta} X \partial x (\cos.\Phi + \sqrt{-1}.\sin.\Phi)}{-2f\sqrt{-1}.\sin.\theta}$$
$$+ \frac{x^{-f\cos.\theta}(\cos.\Phi + \sqrt{-1}.\sin.\Phi)\int x^{f\cos.\theta} X \partial x (\cos.\Phi - \sqrt{-1}.\sin.\Phi)}{2f\sqrt{-1}.\sin.\theta}$$

ubi partes imaginariae se sponte destruunt, fietque

$$Nxy = \frac{x^{-f\cos.\theta}}{f\sin.\theta}(\sin.\Phi\int x^{f\cos.\theta}X\partial x\cos.\Phi - \cos.\Phi\int x^{f\cos.\theta}X\partial x\sin.\Phi).$$

Corollarium 3.

1245. Forma ergo haec realis modo inventa aequivalet illi imaginaria implicanti

$$Nxy = \frac{x^{-\alpha}}{\beta-\alpha}\int x^{\alpha}X\partial x + \frac{x^{-\beta}}{\alpha-\beta}\int x^{\beta}X\partial x,$$

si fuerit

$$(\alpha+z)(\beta+z) = ff + 2fz\cos.\theta + zz,$$

ponaturque $\Phi = f\sin.\theta\, lx$, quae reductio semel facta etiam in sequentibus usum praestabit.

Problema 168.

1246. Proposita hac aequatione differentiali tertii gradus

$$X = Ay + \frac{Bx\partial y}{\partial x} + \frac{Cxx\partial\partial y}{\partial x^2} + \frac{Nx^3\partial^3 y}{\partial x^3},$$

ejus integrale per formulas integrales simplices evoluere.

Solutio.

Cum hic sit

$$P = A + B(z-1) + C(z-1)(z-2) + N(z-1)(z-2)(z-3),$$

ponatur

$$P = N(\alpha+z)(\beta+z)(\gamma+z),$$

et cum per integrationem generalem prodeat $Ny = X'''$, notetur esse $X''' = x^{-\gamma-1}\int x^{\gamma}X''\partial x$, siquidem valorem ipsius X'' ex binis factoribus $\alpha+z$ et $\beta+z$ jam invenimus; hinc enim per problema praecedens habetur

$$X'' = \frac{x^{-\alpha-1}}{\beta-\alpha}\int x^{\alpha}X\partial x + \frac{x^{-\beta-1}}{\alpha-\beta}\int x^{\beta}X\partial x:$$

unde colligitur

$$\int x^{\gamma} X'' \partial x = \frac{x^{\gamma-\alpha}\int x^{\alpha} X \partial x}{(\beta-\alpha)(\gamma-\alpha)} - \frac{\int x^{\gamma} X \partial x}{(\beta-\alpha)(\gamma-\alpha)}$$
$$+ \frac{x^{\gamma-\beta}\int x^{\beta} X \partial x}{(\alpha-\beta)(\gamma-\beta)} - \frac{\int x^{\gamma} X \partial x}{(\alpha-\beta)(\gamma-\beta)}.$$

Est vero

$$\frac{1}{(\beta-\alpha)(\gamma-\alpha)} + \frac{1}{(\alpha-\beta)(\gamma-\beta)} = \frac{-1}{(\alpha-\gamma)(\beta-\gamma)}$$

quod quemadmodum cum per se liquet, tum vero ex Theoremate §. 1169. demonstrato perspicitur. Quocirca integrale quaesitum ita obtinetur expressum

$$N x y = \frac{x^{-\alpha}\int x^{\alpha} X \partial x}{(\beta-\alpha)(\gamma-\alpha)} + \frac{x^{-\beta}\int x^{\beta} X \partial x}{(\alpha-\beta)(\gamma-\beta)} + \frac{x^{-\gamma}\int x^{\gamma} X \partial x}{(\alpha-\gamma)(\beta-\gamma)}.$$

Corollarium 1.

1247. Si forma P duos habeat factores aequales, ut sit $\beta = \alpha$, quia tum est

$$X'' = x^{-\alpha-1}\int \frac{\partial x}{x}\int x^{\alpha} X \partial x, \text{ erit}$$

$$x^{\gamma} X'' \partial x = \frac{x^{\gamma-\alpha}}{\gamma-\alpha}\int \frac{\partial x}{x}\int x^{\alpha} X \partial x - \frac{1}{\gamma-\alpha}\int x^{\gamma-\alpha-1}\partial x \int x^{\alpha} X \partial x, \text{ at}$$

$$\int x^{\gamma-\alpha-1}\partial x \int x^{\alpha} X \partial x = \frac{x^{\gamma-\alpha}}{\gamma-\alpha}\int x^{\alpha} X \partial x - \frac{1}{\gamma-\alpha}\int x^{\gamma} X \partial x;$$

unde colligitur

$$N x y = \frac{x^{-\alpha}}{\gamma-\alpha}\int \frac{\partial x}{x}\int x^{\alpha} X \partial x - \frac{x^{-\alpha}}{(\gamma-\alpha)^2}\int x^{\alpha} X \partial x + \frac{x^{-\gamma}}{(\gamma-\alpha)^2}\int x^{\gamma} X \partial x.$$

Corollarium 2.

1248. Similis forma oritur, si sumatur $\gamma = \beta$, tum enim fit

$$\int x^\gamma X'' \partial x = \frac{x^{\beta-\alpha}}{(\beta-\alpha)^2}\int x^\alpha X \partial x - \frac{\int x^\beta X \partial x}{(\beta-\alpha)^2} + \frac{1}{\alpha-\beta}\int \frac{\partial x}{x}\int x^\beta X \partial x$$

ideoque

$$N.xy = \frac{x^{-\beta}}{\alpha-\beta}\int \frac{\partial x}{x}\int x^\beta X \partial x - \frac{x^{-\beta}\int x^\beta X \partial x}{(\beta-\alpha)^2} + \frac{x^{-\alpha}\int x^\alpha X \partial x}{(\beta-\alpha)^2}.$$

Corollarium 3.

1249. Quodsi autem omnes tres factores inter se fuerint aequales $\alpha = \beta = \gamma$, erit

$$\int x^\gamma X'' \partial x = \int \frac{\partial x}{x}\int \frac{\partial x}{x}\int x^\alpha X \partial x;$$

ideoque hoc casu integrale ita succincte exprimitur

$$N\, x\, y = x^{-\alpha}\int \frac{\partial x}{x}\int \frac{\partial x}{x}\int x^\alpha X \partial x.$$

Corollarium 4.

1250. Si duo factores sint imaginarii, scilicet

$$(\alpha + z)(\beta + z) = ff + 2fz\cos.\theta + zz, \text{ ob}$$

$$\alpha = f(\cos.\theta + \sqrt{-1}.\sin.\theta) \text{ et}$$

$$\beta = f(\cos.\theta - \sqrt{-1}.\sin.\theta),$$

postremum quidem nostri integralis membrum manet reale ob

$$(\alpha - \gamma)(\beta - \gamma) = \gamma\gamma - 2\gamma f\cos.\theta + ff,$$

at bina priora fient, posito $\Phi = f\sin.\theta\, l x$,

$$\frac{x^{-f\cos.\theta}(\cos.\Phi - \sqrt{-1}.\sin.\Phi)\int x^{f\cos.\theta} X \partial x(\cos.\Phi + \sqrt{-1}.\sin.\Phi)}{-2f\sqrt{-1}.\sin.\theta\,[\gamma - f(\cos.\theta + \sqrt{-1}.\sin.\theta)]}$$

$$\frac{+x^{-f\cos.\theta}(\cos.\Phi + \sqrt{-1}.\sin.\Phi)\int x^{f\cos.\theta} X \partial x(\cos.\Phi - \sqrt{-1}.\sin.\Phi)}{2f\sqrt{-1}.\sin.\theta\,[\gamma - f(\cos.\theta - \sqrt{-1}.\sin.\theta)]},$$

quae reducuntur ad hanc formam realem

$$\frac{x^{-f\cos.\theta}[\gamma\,\mathrm{si}.\Phi - f\,\mathrm{si}(\theta+\Phi)]\int x^{f\cos.\theta}X\partial x\,\mathrm{cs}.\Phi - x^{-f\cos.\theta}[\gamma\,\mathrm{cs}.\Phi - f\,\mathrm{cs}.(\theta+\Phi)]\int x^{f\cos.\theta}X\partial x\,\mathrm{si}.\Phi}{f\sin.\theta(\gamma\gamma - 2\gamma f\cos.\theta + ff)}$$

Scholion,

1251. Quod ad factores imaginarios attinet, integralium inde natorum reductio facilius in genere instituetur, unde in his differentialium gradibus determinatis, ei non amplius immorabor. Factores autem aequales hic data opera pro singulis gradibus accuratius persequi est visum, quia supra nimis cito ad evolutionem generalem properanti in insignem errorem illabi contigit, quem statim feliciter evitassem, si eadem methodo ibi essem usus. Hujusmodi autem vitium circa factores imaginarios hic non est pertimescendum, cum in hoc negotio nihil sub specie infinite parvi negligendum occurrat. Ex hoc autem fonte errores illi, quos supra commisi, sunt nati: quod vitium subtile quo clarius ob oculos ponatur, una cum necessaria emendatione hic evoluam. Quaestio scilicet pro casu praesenti huc redit, ut valor harum duarum formularum

$$\frac{x^{-\alpha}\int x^{\alpha}X\partial x}{(\beta-\alpha)(\gamma-\alpha)} + \frac{x^{-\beta}\int x^{\beta}X\partial x}{(\alpha-\beta)(\gamma-\beta)}$$

definiatur, casu quo $\beta = \alpha$ et ambo membra in infinitum excrescent; hunc in finem pono $\beta = \alpha + \omega$, existente ω particula evanescente, et cum sit

$$x^{\beta} = x^{\alpha}x^{\omega} = x^{\alpha}e^{\omega lx} = x^{\alpha}(1 + \omega lx),$$

hincque

$$x^{-\beta} = x^{-\alpha}(1 - \omega lx),$$

habebimus

$$\frac{x^{-\alpha}\int x^{\alpha}X\partial x}{\omega(\gamma-\alpha)} - \frac{x^{-\alpha}(1-\omega lx)\int x^{\alpha}X\partial x(1+\omega lx)}{\omega(\gamma-\beta)}.$$

Quia nunc est

$$\frac{1}{\gamma-\alpha} = \frac{1}{\gamma-\beta+\omega} = \frac{1}{\gamma-\beta} - \frac{\omega}{(\gamma-\beta)^2},$$

prius membrum induit hanc formam

$$\frac{x^{-\alpha}\int x^{\alpha}X\partial x}{\omega(\gamma-\beta)}-\frac{x^{-\alpha}\int x^{\alpha}X\partial x}{(\gamma-\beta)^2};$$

posterius vero evolutum istam

$$\frac{-x^{-\alpha}\int x^{\alpha}X\partial x}{\omega(\gamma-\beta)}+\frac{x^{-\alpha}lx\int x^{\alpha}X\partial x-x^{-\alpha}\int x^{\alpha}X\partial x\,lx}{\gamma-\beta},$$

sicque valor quaesitus casu $\beta=\alpha$ concluditur

$$\frac{x^{-\alpha}(lx\int x^{\alpha}X\partial x-\int x^{\alpha}X\partial x\,lx)}{\gamma-\alpha}-\frac{x^{-\alpha}\int x^{\alpha}X\partial x}{(\gamma-\alpha)^2},\text{ seu}$$

$$\frac{x^{-\alpha}}{\gamma-\alpha}\int\frac{\partial x}{x}\int x^{\alpha}X\partial x-\frac{x^{-\alpha}}{(\gamma-\alpha)^2}\int x^{\alpha}X\partial x,$$

cujus formulae posterius membrum, quod in vitiosa illa methodo erat omissnm, inde resultat, quod hic ad discrimen inter expressiones $\gamma-\alpha$ et $\gamma-\beta$ respeximus, quam necessariam cautionem supra negleximus.

Problema 169.

1252. Proposita hac aequatione differentiali quarti gradus

$$X=Ay+\frac{Bx\partial x}{\partial x}+\frac{Cxx\partial\partial y}{\partial x^2}+\frac{Dx^3\partial^3 y}{\partial x^3}+\frac{Nx^4\partial^4 y}{\partial x^4},$$

ejus integrale per formulas integrales simplices evoluere.

Solutio.

Formata hinc expressione algebraica

$$P=A+B(z-1)+C(z-1)(z-2)+D(z-1)(z-2)(z-3)$$
$$+N(z-1)(z-2)(z-3)(z-4),$$

statuatur

$$P=N(\alpha+z)(\beta+z)(\gamma+z)(\delta+z),$$

et per praecepta generalia est

$$N y = X^{IV}, \text{ existente } X^{IV} = x^{-\delta-1} \int X''' x^{\delta} \partial x,$$

siquidem X''' ex tribus prioribus factoribus determinetur, quemadmodum in problemate praecedente est factum. Valorem scilicet ibi pro $N x y$ inventum hic per $x^{\delta-1} \partial x$ multiplicari oportet, unde oritur

$$\int x^{\delta} X''' \partial x = + \frac{x^{\delta-\alpha} \int x^{\alpha} X \partial x}{(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)} - \frac{\int x^{\delta} X \partial x}{(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)}$$
$$+ \frac{x^{\delta-\beta} \int x^{\beta} X \partial x}{(\alpha-\beta)(\gamma-\beta)(\delta-\beta)} - \frac{\int x^{\delta} X \partial x}{(\alpha-\beta)(\gamma-\beta)(\delta-\beta)}$$
$$+ \frac{x^{\delta-\gamma} \int x^{\gamma} X \partial x}{(\alpha-\gamma)(\beta-\gamma)(\delta-\gamma)} - \frac{\int x^{\delta} X \partial x}{(\alpha-\gamma)(\beta-\gamma)(\delta-\gamma)}$$

ubi ob rationes supra demonstratas tres postremi termini coalescunt in $+ \frac{\int x^{\delta} X \partial x}{(\alpha-\delta)(\beta-\delta)(\gamma-\delta)}$, ita ut sit integrale quaesitum

$$N x y = \frac{x^{-\alpha} \int x^{\alpha} X \partial x}{(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)} + \frac{x^{-\beta} \int x^{\beta} X \partial x}{(\alpha-\beta)(\gamma-\beta)(\delta-\beta)}$$
$$+ \frac{x^{-\gamma} \int x^{\gamma} X \partial x}{(\alpha-\gamma)(\beta-\gamma)(\delta-\gamma)} + \frac{x^{-\delta} \int x^{\delta} X \partial x}{(\alpha-\delta)(\beta-\delta)(\gamma-\delta)}$$

siquidem omnes factores sint inter se inaequales. Casus autem quibus duo pluresve sunt aequales, in corollariis explorabimus.

Corollarium 1.

1253. Si fuerint duo factores aequales nempe $\delta = \gamma$, seu si sit

$$P = N(\alpha + z)(\beta + z)(\gamma + z)^2$$

ex eadem forma pro X''' ante inventa oritur integrale

$$Nxy = \frac{x^{-\alpha}\int x^{\alpha} X \partial x}{(\beta-\alpha)(\gamma-\alpha)^2} - \frac{x^{-\gamma}\int x^{\gamma} X \partial x}{(\beta-\alpha)(\gamma-\alpha)^2}$$
$$+ \frac{x^{-\beta}\int x^{\beta} X \partial x}{(\alpha-\beta)(\gamma-\beta)^2} - \frac{x^{-\gamma}\int x^{\gamma} X \partial x}{(\alpha-\beta)(\gamma-\beta)^2}$$
$$+ \frac{x^{-\gamma}\int \frac{\partial x}{x}\int x^{\gamma} X \partial x}{(\alpha-\gamma)(\beta-\gamma)};$$

ubi membra negativa ita repraesentari possunt

$$\frac{x^{-\gamma}\int x^{\gamma} X \partial x}{(\alpha-\beta)}\left(\frac{1}{(\gamma-\alpha)^2} - \frac{1}{(\gamma-\beta)^2}\right).$$

Corollarium 2.

1254. Si fuerint tres factores aequales, ut sit

$$P = N(\alpha+z)(\beta+z)^3,$$

ideoque $\delta = \gamma = \beta$, ex formula §. 1248. inventa colligitur integrale

$$Nxy = \frac{x^{-\alpha}\int x^{\alpha} X \partial x}{(\beta-\alpha)^3} - \frac{x^{-\beta}\int x^{\beta} X \partial x}{(\beta-\alpha)^3}$$
$$- \frac{x^{-\beta}\int \frac{\partial x}{x}\int x^{\beta} X \partial x}{(\beta-\alpha)^2} + \frac{x^{-\beta}\int \frac{\partial x}{x}\int \frac{\partial x}{x}\int x^{\beta} X \partial x}{\alpha-\beta}.$$

Corollarium 3.

1255. Si omnes quatuor factores fuerint aequales, ut sit

$P = N(\alpha+z)^4$, existente $\delta = \gamma = \beta = \alpha$,

ex forma pro tribus aequalibus §. 1249. inventa fit integrale

$$Mxy = x^{-\alpha}\int \frac{\partial x}{x}\int \frac{\partial x}{x}\int \frac{\partial x}{x}\int x^{\alpha} X \partial x.$$

Corollarium 4.

1256. Si habeatur $\beta = \alpha$ et $\delta = \gamma$, ut bini factores sint aequales scilicet $P = N(\alpha + z)^2 (\gamma + z)^2$, ex §. 1247, ubi factores erant $(\alpha + z)^2 (\gamma + z)$, colligitur integrale

$$Nxy = \frac{x^{-\alpha}}{(\gamma-\alpha)^2}\int\frac{\partial x}{x}\int x^\alpha X\partial x - \frac{x^{-\gamma}}{(\gamma-\alpha)^2}\int x^{\gamma-\alpha-1}\partial x\int x^\alpha X\partial x$$
$$- \frac{x^{-\alpha}}{(\gamma-\alpha)^3}\int x^\alpha X\partial x + \frac{x^{-\gamma}}{(\gamma-\alpha)^3}\int x^\gamma X\partial x + \frac{x^{-\gamma}}{(\gamma-\alpha)^2}\int\frac{\partial x}{x}\int x^\gamma X\partial x;$$

quae ob

$$\int x^{\gamma-\alpha-1}\partial x\int x^\alpha X\partial x = \frac{x^{\gamma-\alpha}}{\gamma-\alpha}\int x^\alpha X\partial x - \frac{1}{\gamma-\alpha}\int x^\gamma X\partial x$$

contrahitur in hanc formam

$$Nxy = \frac{x^{-\alpha}}{(\gamma-\alpha)^2}\int\frac{\partial x}{x}\int x^\alpha X\partial x + \frac{x^{-\gamma}}{(\gamma-\alpha)^2}\int\frac{\partial x}{x}\int x^\gamma X\partial x$$
$$- \frac{2x^{-\alpha}}{(\gamma-\alpha)^3}\int x^\alpha X\partial x - \frac{2x^{-\gamma}}{(\alpha-\gamma)^3}\int x^\gamma X\partial x.$$

Problema 170.

1257. Proposita hac aequatione differentiali quinti gradus

$$X = Ay + \frac{Bx\partial y}{\partial x} + \frac{Cx^2\partial\partial y}{\partial x^2} + \frac{Dx^3\partial^3 y}{\partial x^3} + \frac{Ex^4\partial^4 y}{\partial x^4} + \frac{Nx^5\partial^5 y}{\partial x^5},$$

ejus integrale per formulas integrales simplices evoluere.

Solutio.

Cum hic sit quantitas algebraica formanda

$$P = A + B(z-1) + C(z-1)(z-2) + \ldots\ldots$$
$$\ldots + N(z-1)(z-2)(z-3)(z-4)(z-5)$$

statuatur

$$P = N(\alpha + z)(\beta + z)(\gamma + z)(\delta + z)(\varepsilon + z),$$

ac si hi factores omnes sint inter se inaequales, ex integrali praecedente novam instituendo integrationem prodibit integrale quaesitum

$$\begin{aligned} Nxy = & \frac{x^{-\alpha}\int x^{\alpha} X \partial x}{(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)(\varepsilon-\alpha)} + \frac{x^{-\beta}\int x^{\beta} X \partial x}{(\alpha-\beta)(\gamma-\beta)(\delta-\beta)(\varepsilon-\beta)} \\ & + \frac{x^{-\gamma}\int x^{\gamma} X \partial x}{(\alpha-\gamma)(\beta-\gamma)(\delta-\gamma)(\varepsilon-\gamma)} + \frac{x^{-\delta}\int x^{\delta} X \partial x}{(\alpha-\delta)(\beta-\delta)(\gamma-\delta)(\varepsilon-\delta)} \\ & + \frac{x^{-\varepsilon}\int x^{\varepsilon} X \partial x}{(\alpha-\varepsilon)(\beta-\varepsilon)(\gamma-\varepsilon)(\delta-\varepsilon)}. \end{aligned}$$

Casus quo duo pluresve factores sunt aequales, in corollariis evoluemus.

Corollarium 1.

1258. Si fuerint duo factores aequales, ut sit

$P = N(\alpha + z)(\beta + z)(\gamma + z)(\delta + z)^2$, ideoque $\varepsilon = \delta$

ex praecedente problemate colligitur integrale

$$\begin{aligned} Nxy = & + \frac{x^{-\alpha}\int x^{\alpha} X \partial x - x^{-\delta}\int x^{\delta} X \partial x}{(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)^2} \\ & + \frac{x^{-\beta}\int x^{\beta} X \partial x - x^{-\delta}\int x^{\delta} X \partial x}{(\alpha-\beta)(\gamma-\beta)(\delta-\beta)^2} \\ & + \frac{x^{-\gamma}\int x^{\gamma} X \partial x - x^{-\delta}\int x^{\delta} X \partial x}{(\alpha-\gamma)(\beta-\gamma)(\delta-\gamma)^2} \\ & + \frac{x^{-\delta}\int \frac{\partial x}{x}\int x^{\delta} X \partial x}{(\alpha-\delta)(\beta-\delta)(\gamma-\delta)}. \end{aligned}$$

Corollarium 2.

1259. Si fuerint tres fa tores aequales, ut sit

$P = N(\alpha + z)(\beta + z)(\gamma + z)^3$, ideoque

$\varepsilon = \delta = \gamma$

ex corollario 1. problematis praecedentis colligitur

$$N x y = \frac{x^{-\alpha}\int x^\alpha X \partial x - x^{-\gamma}\int x^\gamma X \partial x}{(\beta - \alpha)(\gamma - \alpha)^3} - \frac{x^{-\gamma}\int \frac{\partial x}{x}\int x^\gamma X \partial x}{(\beta - \alpha)(\gamma - \alpha)^2}$$

$$+ \frac{x^{-\beta}\int x^\beta X \partial x - x^{-\gamma}\int x^\gamma X \partial x}{(\alpha - \beta)(\gamma - \beta)^3} - \frac{x^{-\gamma}\int \frac{\partial x}{x}\int x^\gamma X \partial x}{(\alpha - \beta)(\gamma - \beta)^2}$$

$$+ \frac{x^{-\gamma}\int \frac{\partial x}{x}\int \frac{\partial x}{x}\int x^\gamma X \partial x}{(\alpha - \gamma)(\beta - \gamma)}.$$

Corollarium 3.

1260. Si quatuor factores sint aequales, ut sit

$P = N(\alpha + z)(\beta + z)^4$, ideoque

$\varepsilon = \delta = \gamma = \beta$, erit per §. 1254

$$N x y = \frac{x^{-\alpha}\int x^\alpha X \partial x}{(\beta - \alpha)^4} - \frac{x^{-\beta}\int x^\beta X \partial x}{(\beta - \alpha)^4} - \frac{x^{-\beta}\int \frac{\partial x}{x}\int x^\beta X \partial x}{(\beta - \alpha)^3}$$

$$- \frac{x^{-\beta}\int \frac{\partial x}{x}\int \frac{\partial x}{x}\int x^\beta X \partial x}{(\beta - \alpha)^2} + \frac{x^{-\beta}\int \frac{\partial x}{x}\int \frac{\partial x}{x}\int \frac{\partial x}{x}\int x^\beta X \partial x}{\alpha - \beta}.$$

Ac si omnes quinque sint inter se aequales seu

$P = N(\alpha + z)^5$, erit integrale

$$N x y = x^{-\alpha}\int \frac{\partial x}{x}\int \frac{\partial x}{x}\int \frac{\partial x}{x}\int \frac{\partial x}{x}\int x^\alpha X \partial x.$$

Corollarium 4.

1261. Si P habeat duos factores quadratos, ut sit

$P = N(\alpha + z)(\beta + z)^2(\gamma + z)^2$, ideoque $\delta = \gamma$ et $\varepsilon = \beta$,

erit ex §. 1253. integrale, reductione necessaria facta,

$$Nxy = \frac{x^{-\alpha}\int x^{\alpha}X\partial x - x^{-\beta}x^{\beta}X\partial x}{(\beta-\alpha)^2(\gamma-\alpha)^2} - \frac{x^{-\gamma}x^{\gamma}X\partial x + x^{-\beta}x^{\beta}X\partial x}{(\beta-\alpha)(\alpha-\gamma)^2(\beta-\gamma)}$$
$$+ \frac{x^{-\beta}\int\frac{\partial x}{x}\int x^{\beta}X\partial x}{(\alpha-\beta)(\gamma-\beta)^3} - \frac{x^{-\gamma}\int x^{\gamma}X\partial x + x^{-\beta}\int x^{\beta}X\partial x}{(\alpha-\beta)(\beta-\gamma)^3}$$
$$+ \frac{x^{-\gamma}\int\frac{\partial x}{x}\int x^{\gamma}X\partial x}{(\alpha-\gamma)(\beta-\gamma)^2} - \frac{x^{-\gamma}\int x^{\gamma}X\partial x + x^{-\beta}\int x^{\beta}X\partial x}{(\alpha-\gamma)(\beta-\gamma)^3},$$

quae porro redigitur ad hanc formam

$$Nxy = \frac{x^{-\alpha}\int x^{\alpha}X\partial x}{(\beta-\alpha)^2(\gamma-\alpha)^2} + \frac{x^{-\beta}\int\frac{\partial x}{x}\int x^{\beta}X\partial x}{(\alpha-\beta)^2(\gamma-\beta)^2} - \frac{x^{-\beta}\int x^{\beta}X\partial x}{(\alpha-\beta)^2(\gamma-\beta)^2} - \frac{2x^{-\beta}\int x^{\beta}X\partial x}{(\alpha-\beta)(\gamma-\beta)^3}$$
$$+ \frac{x^{-\gamma}\int\frac{\partial x}{x}\int x^{\gamma}X\partial x}{(\alpha-\gamma)(\beta-\gamma)^2} - \frac{x^{-\gamma}\int x^{\gamma}X\partial x}{(\alpha-\gamma)^2(\beta-\gamma)^2} - \frac{2x^{-\gamma}\int x^{\gamma}X\partial x}{(\alpha-\gamma)(\beta-\gamma)^3}.$$

Corollarium 5.

1262. Si P habeat et factorem quadratum et cubicum, ut sit $P = N(\alpha+z)^2(\gamma+z)^3$, ideoque $\beta = \alpha$ et $\varepsilon = \delta = \gamma$, ex §. 1254. colligitur integrale

$$Nxy = \frac{x^{-\alpha}\int\frac{\partial x}{x}\int x^{\alpha}X\partial x}{(\gamma-\alpha)^3} - \frac{3x^{-\alpha}\int x^{\alpha}X\partial x}{(\gamma-\alpha)^4}$$
$$+ \frac{x^{-\gamma}\int\frac{\partial x}{x}\int\frac{\partial x}{x}\int x^{\gamma}X\partial x}{(\alpha-\gamma)^2} - \frac{2x^{-\gamma}\int\frac{\partial x}{x}\int x^{\gamma}X\partial x}{(\alpha-\gamma)^3} - \frac{3x^{-\gamma}\int x^{\gamma}X\partial x}{(\alpha-\gamma)^4}.$$

Scholion.

1263. Ex his formulis parum constat, quemadmodum eas ulterius pro majori factorum numero continuari oportet, si quidem factorum aliquot inter se fuerint aequales; integralium enim partes,

quae factoribus inaequalibus respondent, legem servant manifestam. Quae autem partibus aequalibus respondent, adhibita certa reductione commodius exprimi possunt. Veluti pro casu corollarii 1. si brevitatis gratia ponatur $\alpha - \delta = p$, $\beta - \delta = q$ et $\gamma - \delta = r$, forma $x^{-\delta} \int x^{\delta} X \partial x$ ducta est in

$$\frac{1}{(p-q)(r-p)pp} + \frac{1}{(p-q)(q-r)qq} + \frac{1}{(r-p)(q-r)rr}, \text{ seu}$$
$$\frac{(q-r)qqrr + (r-p)pprr + (p-q)ppqq}{(p-q)(q-r)(r-p)ppqqrr},$$

cujus fractionis numerator est

$$-(p-q)(q-r)(r-p)(pq+pr+qr),$$

ita ut haec fractio reducatur ad istam

$$\frac{-pq-pr-qr}{ppqqrr} = -\frac{1}{pqr}\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right)$$

Quando ergo est

$$P = N(\alpha+z)(\beta+z)(\gamma+z)(\delta+z)^2,$$

integrale ita se habet

$$Nxy = \frac{x^{-\alpha}\int x^{\alpha} X \partial x}{(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)^2} + \frac{x^{-\beta}\int x^{\beta} X \partial x}{(\alpha-\beta)(\gamma-\beta)(\delta-\beta)^2} + \frac{x^{-\gamma}\int x^{\gamma} X \partial x}{(\alpha-\gamma)(\beta-\gamma)(\delta-\gamma)^2}$$
$$+ \frac{x^{-\delta}\int \frac{\partial x}{x} x^{\delta} X \partial x}{(\alpha-\delta)(\beta-\delta)(\gamma-\delta)} - \frac{x^{-\delta}\int x^{\delta} X \partial x}{(\alpha-\delta)(\beta-\delta)(\gamma-\delta)}\left(\frac{1}{\alpha-\delta}+\frac{1}{\beta-\delta}+\frac{1}{\gamma-\delta}\right).$$

Pro casu autem $P = M(\alpha+z)(\beta+z)(\gamma+z)^3$ habebitur

$$Nxy = \frac{x^{-\alpha}\int x^{\alpha} X \partial x}{(\beta-\alpha)(\gamma-\alpha)^3} + \frac{x^{-\beta}\int x^{\beta} X \partial x}{(\alpha-\beta)(\gamma-\beta)^3}$$
$$+ \frac{x^{-\gamma}\int \frac{\partial x}{x}\int \frac{\partial x}{x}\int x^{\gamma} X \partial x}{(\alpha-\gamma)(\beta-\gamma)} - \frac{x^{-\gamma}\int \frac{\partial x}{x}\int x^{\gamma} X \partial x}{(\alpha-\gamma)(\beta-\gamma)}\left(\frac{1}{\alpha-\gamma}+\frac{1}{\beta-\gamma}\right)$$
$$+ \frac{x^{-\gamma}\int x^{\gamma} X \partial x}{(\alpha-\gamma)(\beta-\gamma)}\left(\frac{1}{(\alpha-\gamma)^2}+\frac{1}{(\alpha-\gamma)(\beta-\gamma)}+\frac{1}{(\beta-\gamma)^2}\right).$$

Tum vero pro casu $P = N(\alpha + z)(\beta + z)^4$ fit

$$Nxy = \frac{x^{-\alpha}\int x^\alpha X\,\partial x}{(\beta-\alpha)^4} + \frac{x^{-\beta}\int\frac{\partial x}{x}\int\frac{\partial x}{x}\int\frac{\partial x}{x}\int x^\beta X\,\partial x}{\alpha-\beta}$$

$$-\frac{x^{-\beta}\int\frac{\partial x}{x}\int\frac{\partial x}{x}\int x^\beta X\,\partial x}{\alpha-\beta}\cdot\frac{1}{\alpha-\beta} + \frac{x^{-\beta}\int\frac{\partial x}{x}\int x^\beta X\,\partial x}{\alpha-\beta}\cdot\frac{1}{(\alpha-\beta)^2}$$

$$-\frac{x^{-\beta}\int x^\beta X\,\partial x}{\alpha-\beta}\cdot\frac{1}{(\alpha-\beta)^3}.$$

At pro casu $P = (\alpha + z)(\beta + z)^2(\gamma + z)^2$ integrale ita se habet

$$Nxy = \frac{x^{-\alpha}\int x^\alpha X\,\partial x}{(\beta-\alpha)^2(\gamma-\alpha)^2} + \frac{x^{-\beta}\int\frac{\partial x}{x}\int x^\beta X\,\partial x}{(\alpha-\beta)(\gamma-\beta)^2}$$

$$-\frac{x^{-\beta}\int x^\beta X\,\partial x}{(\alpha-\beta)(\gamma-\beta)^2}\left(\frac{1}{\alpha-\beta}+\frac{2}{\gamma-\beta}\right) + \frac{x^{-\gamma}\int\frac{\partial x}{x}\int x^\gamma X\,\partial x}{(\alpha-\gamma)(\beta-\gamma)^2}$$

$$-\frac{x^{-\gamma}\int x^\gamma X\,\partial x}{(\alpha-\gamma)(\beta-\gamma)^2}\left(\frac{1}{\alpha-\gamma}+\frac{2}{\beta-\gamma}\right).$$

Denique pro casu $P = N(\alpha + z)^2(\gamma + z)^3$ est

$$Nxy = \frac{x^{-\alpha}\int\frac{\partial x}{x}\int x^\alpha X\,\partial x}{(\gamma-\alpha)^3} - \frac{x^{-\alpha}\int x^\alpha X\,\partial x}{(\gamma-\alpha)^3}\cdot\frac{3}{\gamma-\alpha}$$

$$+\frac{x^{-\gamma}\int\frac{\partial x}{x}\int\frac{\partial x}{x}\int x^\gamma X\partial x}{(\alpha-\gamma)^2} - \frac{x^{-\gamma}\int\frac{\partial x}{x}\int x^\gamma X\partial x}{(\alpha-\gamma)^2}\cdot\frac{2}{\alpha-\gamma}$$

$$+\frac{x^{-\gamma}\int x^\gamma X\,\partial x}{(\alpha-\gamma)^2}\cdot\frac{3}{(\alpha-\gamma)^2},$$

unde indoles harum formularum jam magis fit perspicua, simulque patet partem integralis ex aliquot factoribus oriundam non pendere ab aequalitate reliquorum. Quocirca jam problema generale aggredi licebit.

Problema 171.

1264. Proposita aequatione differentiali cujuscunque gradus hujus formae

$$X = Ay + \frac{Bx\partial y}{\partial x} + \frac{Cx^2\partial\partial y}{\partial x^2} + \frac{Dx^3\partial^3 y}{\partial x^3} + \ldots + \frac{Nx^n\partial^n y}{\partial x^n},$$

ex qua forma algebraica hac lege formata

$$P = A + B(z-1) + C(z-1)(z-2) + D(z-1)(z-2)(z-3) + \ldots$$
$$\ldots + N(z-1)(z-2)\ldots\ldots(z-n),$$

omnes factores habeat inter se inaequales; valorem ipsius y completum per formulas integrales simplices exhibere.

Solutio.

Sint primo formae P omnes factores simplices reales

$$P = N(\alpha+z)(\beta+z)(\gamma+z)\ \ldots\ldots\ (\nu+z),$$

factorum numero existente $= n$, et ex antecedentibus patet, ex quolibet factore nasci integralis partem. Ad has partes inveniendas, eliciantur sequentes valores

1.) posito $z = -\alpha$ sit $\mathfrak{A} = \frac{P}{\alpha+z}$, seu $\mathfrak{A} = \frac{\partial P}{\partial z}$,

2.) posito $z = -\beta$ sit $\mathfrak{B} = \frac{P}{\beta+z}$, seu $\mathfrak{B} = \frac{\partial P}{\partial z}$,

3.) posito $z = -\gamma$ sit $\mathfrak{C} = \frac{P}{\gamma+z}$, seu $\mathfrak{C} = \frac{\partial P}{\partial z}$,

etc.

Cum igitur sit

$$(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)\ldots\ldots(\nu-\alpha) = \frac{\mathfrak{A}}{N},$$

litterâ N ex superioribus formis per divisionem tolletur, fietque integrale quaesitum

$$xy = \frac{1}{\mathfrak{A}}x^{-\alpha}\int x^{\alpha}X\partial x + \frac{1}{\mathfrak{B}}x^{-\beta}\int x^{\beta}X\partial x + \frac{1}{\mathfrak{C}}x^{-\gamma}\int x^{\gamma}X\partial x + \text{etc.}$$

quoad singuli factores fuerint exhausti.

Quodsi jam forma P factores habeat imaginarios, partium inde ortarum imaginariarum ad realitatem reductio sequenti modo instituetur. Quoniam bini factores simplices imaginarii praebent factorem duplicem realem, ponamus

$$(\alpha + z)(\beta + z) = ff + 2fz\cos.\theta + zz,$$

ita ut sit

$$\alpha = f(\cos.\theta + \surd - 1.\sin\theta) \text{ et } \beta = f(\cos.\theta - \surd - 1.\sin\theta),$$

unde primum valores literarum $\mathfrak{A}$ et $\mathfrak{B}$ definiantur, quarum cum utraque derivetur ex forma $\frac{\partial P}{\partial z}$, illa posito $z = -\alpha$ haec vero posito $z = -\beta$, in ipsa forma $\frac{\partial P}{\partial z}$ loco z ubique scribatur

$$-f(\cos.\theta \pm \surd - 1.\sin.\theta),$$

prodeatque $\mathfrak{P} \pm \mathfrak{Q}\surd - 1$. Ac perspicuum est fore

$$\mathfrak{A} = \mathfrak{P} + \mathfrak{Q}\surd - 1 \text{ et } \mathfrak{B} = \mathfrak{P} - \mathfrak{Q}\surd - 1,$$

ubi notandum est, quantitates $\mathfrak{P}$ et $\mathfrak{Q}$ esse reales. Deinde cum sit

$$x^{m+n\surd-1} = x^m e^{n\surd-1.lx} = x^m[\cos.(nlx) + \surd - 1.\sin.(nlx)],$$

si brevitatis ergo ponamus angulum $f\sin.\theta\, lx = \Phi$, erit

$$x^\alpha = x^{f\cos.\theta}(\cos.\Phi + \surd - 1.\sin.\Phi),$$
$$x^{-\alpha} = x^{-f\cos.\theta}(\cos.\Phi - \surd - 1.\sin.\Phi),$$
$$x^\beta = x^{f\cos.\theta}(\cos.\Phi - \surd - 1.\sin.\Phi),$$
$$x^{-\beta} = x^{-f\cos.\theta}(\cos.\Phi + \surd - 1.\sin.\Phi).$$

Quare pro binis partibus

$$\frac{1}{\mathfrak{A}}x^{-\alpha}\int x^\alpha X\partial x + \frac{1}{\mathfrak{B}}x^{-\beta}\int x^\beta X\partial x,$$

ob $\mathfrak{A}\mathfrak{B} = \mathfrak{P}\mathfrak{P} + \mathfrak{Q}\mathfrak{Q}$ habebimus

$$+\frac{x^{-f\cos.\theta}}{\mathfrak{P}\mathfrak{P}+\mathfrak{Q}\mathfrak{Q}}\left\{\begin{array}{l}(\mathfrak{P}-\mathfrak{Q}\surd-1)(\cos.\Phi-\surd-1.\sin.\Phi)\int x^{f\cos.\theta}(\cos.\Phi+\surd-1.\sin\Phi)X\partial x\\(\mathfrak{P}+\mathfrak{Q}\surd-1)(\cos.\Phi+\surd-1.\sin.\Phi)\int x^{f\cos\theta}(\cos.\Phi-\surd-1.\sin.\Phi)X\partial x\end{array}\right\}$$

quae forma ob partes imaginarias se tollentes reducitur ad hanc

$$\frac{\left\{\begin{array}{l} 2\, x^{-f \cos.\theta} (\mathfrak{P} \cos.\Phi - \mathfrak{Q} \sin.\Phi) \int x^{f \cos.\theta} X \partial x \cos.\Phi \\ + 2\, x^{-f \cos\theta} (\mathfrak{Q} \cos.\Phi + \mathfrak{P} \sin.\Phi) \int x^{f \cos.\theta} X \partial x \sin.\Phi \end{array}\right\}}{\mathfrak{P}\mathfrak{P} + \mathfrak{Q}\mathfrak{Q}}.$$

Talisque forma ad integrale accedit, quoties forma P hujusmodi habet factorem duplicem $ff + 2fz \cos.\theta + zz$.

Corollarium 1.

1265. Etsi autem factorum simplicium ipsius P quidam sunt imaginarii, eorum qui sunt reales evolutio inde non perturbatur, sed ex singulis partes in integrale inferendae a natura reliquorum factorum minime pendent.

Corollarium 2.

1266. Pars integralis ex binis factoribus imaginariis seu uno factore duplici oriunda aliquanto succinctius repraesentari potest, si ponatur

$$\mathfrak{P} = \mathfrak{D} \cos.\zeta \quad \text{et} \quad \mathfrak{Q} = \mathfrak{D} \sin.\zeta,$$

sic enim ea fiet

$$\frac{2}{\mathfrak{D}} x^{-f \cos.\theta} [\cos.(\zeta + \Phi) \int x^{f \cos.\theta} X \partial x \cos.\Phi \\ + \sin.(\zeta + \Phi) \int x^{f \cos.\theta} X \partial x \sin.\Phi],$$

ubi ζ et θ sunt anguli constantes, Φ vero variabilis ob $\Phi = f \sin.\theta \,.\, l\, x$.

Problema 172.

1267. Si pro aequatione differentiali in praecedente problemate proposita quantitas algebraica P inde formata duos habeat factores simplices aequales, integralis partem inde oriundam investigare.

Solutio.

In forma ergo ante exhibita

$$P = N(\alpha + z)(\beta + z)(\gamma + z)\,\delta + z)\ \text{etc.}$$

ponamus esse $\beta = \alpha$, quoniam vero tum utraque integralis pars oritur infinita, altera signo $+$, altera signo $-$ affecta, ita ut junctim sumtae partem constituant finitam, ad hanc eliciendam statuamus $\beta = \alpha - \omega$, denotante ω quantitatem evanescentem, eritque

$$\mathfrak{A} = -N\omega(\gamma - \alpha)(\delta - \alpha)(\varepsilon - \alpha)\ \text{etc. et}$$
$$\mathfrak{B} = +N\omega(\gamma - \beta)(\delta - \beta)(\varepsilon + \beta)\ \text{etc.}$$

Ponatur jam

$$\frac{P}{(\alpha + z)(\beta + z)} = \frac{P}{(\alpha + z)^2} = Q,$$

ut sit

$$Q = N(\gamma + z)(\delta + z)(\varepsilon + z)\ \text{etc.}$$

ac manifestum est fieri

$$\mathfrak{A} = -\omega Q,\ \text{posito}\ z = -\alpha\ \text{et}$$
$$\mathfrak{B} = +\omega Q,\ \text{posito}\ z = -\beta = -\alpha + \omega,$$

unde intelligitur valorem ipsius Q posteriorem excedere priorem suo differentiali ∂Q, si fiat

$$z = -\alpha\ \text{et}\ \partial z = \omega,$$

ita ut sit

$$\mathfrak{B} = \omega\left(Q + \omega\frac{\partial Q}{\partial z}\right),\ \text{posito}\ z = -\alpha,$$

hincque

$$\frac{1}{\mathfrak{B}} = \frac{1}{\omega Q} - \frac{\partial Q}{QQ\,\partial z} = \frac{1}{\omega Q} + \frac{1}{\partial z}\,\partial\cdot\frac{1}{Q},\ \text{existente}\ \frac{1}{\mathfrak{A}} = -\frac{1}{\omega Q}:$$

tum vero cum sit

$$x^\beta = x^\alpha x^{-\omega} = x^\alpha(1 - \omega\, l\, x)\ \text{et}\ x^{-\beta} = x^{-\alpha}(1 + \omega\, l\, x),$$

binae partes integralis quaesitae erunt

$$-\frac{1}{\omega Q} x^{-\alpha} \int x^\alpha X \partial x$$
$$+ \left(\frac{1}{\omega Q} + \frac{1}{\partial z} \partial . \frac{1}{Q}\right) x^{-\alpha} (1 + \omega l x) \int x^\alpha X \partial x (1 - \omega l x),$$

ubi cum membra per ω divisa se destruant, resultat

$$\frac{1}{Q} x^{-\alpha} (l x \int x^\alpha X \partial x - \int x^\alpha X \partial x\, l x) + \frac{1}{\partial z} \partial . \frac{1}{Q} . x^{-\alpha} \int x^\alpha X \partial x$$

seu

$$\frac{1}{Q} x^{-\alpha} \int \frac{\partial x}{x} \int x^\alpha X \partial x + \frac{1}{\partial z} \partial . \frac{1}{Q} . x^{-\alpha} \int x^\alpha X \partial x,$$

siquidem tam in valore $\frac{1}{Q}$ quam in $\frac{1}{\partial z} \partial . \frac{1}{Q}$ ubique loco z scribatur $-\alpha$. Cum vero sit $Q = \frac{P}{(\alpha + z)^2}$, hi valores inde facile inveniuntur.

Corollarium 1.

1268. Quodsi ergo quantitas algebraica P ex aequatione differentiali formata factorem habeat quadratum $(\alpha + z)^2$, inde in integrale transferenda est haec portio

$$\frac{(\alpha+z)^2}{P} x^{-\alpha} \int \frac{\partial x}{x} \int x^\alpha X \partial x + \frac{1}{\partial z} \partial . \frac{(\alpha+z)^2}{P} . x^{-\alpha} \int x^\alpha X \partial x,$$

posito $z = -\alpha$; dum si hic factor $\alpha + z$ esset solitarius integralis pars inde oriunda foret

$$\frac{\alpha + z}{P} x^{-\alpha} \int x^\alpha X \partial x, \text{ posito } z = -\alpha.$$

Corollarium 2.

1269. Cum sit $Q = \frac{P}{(\alpha + z)^2}$, casu $z = -\alpha$ fiet $Q = \frac{\partial \partial P}{2 \partial z^2}$; verum quia hic ipsi z jam valor determinatus est tributus, hinc $\frac{\partial Q}{\partial z}$ colligere non licet, sed prima est utendum qua fit $\frac{\partial Q}{\partial z} = \frac{(\alpha + z) \partial P - 2 P \partial z}{(\alpha + z)^3 \partial z}$, cujus fractionis cum numerator et denominator casu $z = -\alpha$ evanescat, erit pro eodem casu

$$\frac{\partial Q}{\partial z} = \frac{(\alpha + z) \partial \partial P - \partial z \partial P}{3 (\alpha + z)^2 \partial z^2} = \frac{(\alpha + z) \partial^3 P}{6 (\alpha + z) \partial z^3} = \frac{\partial^3 P}{6 \partial z^3}.$$

Corollarium 3.

1270. Hoc valore invento, quia est eodem casu $z = -\alpha$, quantitas $Q = \frac{\partial\partial P}{2\partial z^2}$ erit

$$\frac{1}{\partial z}\partial \cdot \frac{1}{Q} = -\frac{\partial Q}{QQ\partial z} = -\frac{2\partial z\partial^3 P}{3\partial\partial P^2}, \text{ seu}$$

$$\frac{1}{\partial z}\partial \cdot \frac{1}{Q} = \frac{2\partial z}{3}\partial \cdot \frac{1}{\partial\partial P},$$

ex quibus formulis, si factores ipsius P non sint evoluti, partes integralis facilius reperiuntur.

Problema 173.

1271. Si pro aequatione differentiali praecedente quantitas algebraica P inde formata factorem habeat cubicum $(\alpha + z)^3$, integralis partem inde oriundam investigare.

Solutio.

Ponamus ergo esse

$$P = (\alpha + z)^2 (\gamma + z) R, \text{ existente } \gamma = \alpha - \omega,$$

ubi ω pro quantitate evanescente assumitur. Quod ergo ante erat Q, id hinc fit $Q = (\gamma + z) R$, et facto $z = -\alpha$, erit $Q = -\omega R$, si etiam in R ponatur $z = -\alpha$. Deinde cum sit

$$\frac{\partial Q}{\partial z} = R + \frac{(\gamma + z)\partial R}{\partial z} = R - \frac{\omega\partial R}{\partial z},$$

eodem casu erit

$$\frac{1}{\partial z}\partial \cdot \frac{1}{Q} = -\frac{1}{\omega\omega R} + \frac{\partial R}{\omega RR\partial z} = -\frac{1}{\omega\omega} \cdot \frac{1}{R} - \frac{1}{\omega\partial z}\partial \cdot \frac{1}{R}.$$

Quocirca ex factore quadrato $(\alpha + z)^2$ per praecedens problema haec obtinetur integralis pars

$$-\frac{1}{\omega R} x^{-\alpha} \int \frac{\partial x}{x} \int x^\alpha X \partial x - \left(\frac{1}{\omega\omega R} + \frac{1}{\omega\partial z}\partial \cdot \frac{1}{R}\right) x^{-\alpha} \int x^\alpha X \partial x$$

cujus ambo membra in infinitum excrescunt ob $\omega = 0$.

Adjiciamus autem partem ex tertio factore

$$\gamma + z = \alpha - \omega + z,$$

oriundam, quae ob $\frac{P}{\gamma+z} = (\alpha+z)^2 R$ est

$$\frac{1}{(\alpha+z)^2 R} x^{-\gamma} \int x^{\gamma} X \partial x, \text{ posito } z = -\gamma = -\alpha + \omega.$$

Quod si jam R ut ante is fuerit valor, qui oritur posito $z = -\alpha$, augendo hunc valorem particula ω, loco $\frac{1}{R}$ scribi debet

$$\frac{1}{R} + \frac{\omega}{\partial z} \partial \cdot \frac{1}{R} + \frac{\omega^2}{1.2 \partial z^2} \partial \partial \cdot \frac{1}{R} \text{ etc.}$$

si quidem valorem $z = -\alpha$ et hic retineamus: unde haec integralis pars ob $\alpha + z = \omega$ erit

$$\left(\frac{1}{\omega\omega R} + \frac{1}{\omega \partial z} \partial \cdot \frac{1}{R} + \frac{1}{2 \partial z^2} \partial \partial \cdot \frac{1}{R}\right) x^{-\alpha+\omega} \int x^{\alpha-\omega} X \partial x;$$

sicque manifestum est, illum ipsius $\frac{1}{R}$ valorem usque ad secundam potestatem ipsius ω continuari debuisse, atque eadem lege hic alteram partem x involuentem exprimi conveniet. Ad quod observo, si habeatur hujusmodi formula $x^{\omega} \int x^{-\omega} V \partial x$ secundum potestates ipsius ω evoluenda, id hac ratione commodissime fieri. Posito

$$v = x^{\omega} \int x^{-\omega} V \partial x, \text{ ut sit } x^{-\omega} v = \int x^{-\omega} V \partial x,$$

erit differentiando $\partial v - \frac{\omega v \partial x}{x} = V \partial x$, quare posito

$$v = T + \omega T' + \omega^2 T'' + \omega^3 T''' + \text{ etc.}$$

habebitur, terminos secundum potestates ipsius ω disponendo,

$$\left.\begin{array}{l} \partial T + \omega \partial T' + \omega\omega \partial T'' + \omega^3 \partial T''' + \text{etc.} \\ -V \partial x - \omega T \frac{\partial x}{x} - \omega\omega T' \frac{\partial x}{x} - \omega^3 T'' \frac{\partial x}{x} - \text{etc.} \end{array}\right\} = 0$$

ideoque

$$T = \int V \partial x, \quad T' = \int \frac{\partial x}{x} \int V \partial x, \quad T'' = \int \frac{\partial x}{x} \int \frac{\partial x}{x} \int V \partial x, \text{ etc.}$$

Consequenter cum in applicatione sit $V = x^{\alpha} X$, erit pars integralis ex factore $\gamma + z = \alpha - \omega + z$ nata

$$\left(\frac{1}{\omega\omega R} + \frac{1}{\omega \partial z} \partial \cdot \frac{1}{R} + \frac{1}{2 \partial z^2} \partial \partial \cdot \frac{1}{R}\right) x^{-\alpha} \left(\int x^{\alpha} X \partial x + \omega \int \frac{\partial x}{x} \int x^{\alpha} X \partial x + \omega^2 \int \frac{\partial x}{x} \int \frac{\partial x}{x} \int x^{\alpha} X \partial x\right),$$

qua cum parte ex $(\alpha + z)^2$ nata junctim sumta, omnia membra infinita se mutuo destruunt, et pro quantitatis $P = (\alpha + z)^3 R$ factore cubico $(\alpha + z)^3$ in integrale ingreditur haec pars

$$\frac{1}{R} x^{-\alpha} \int \frac{\partial x}{x} \int \frac{\partial x}{x} \int x^\alpha X \partial x + \frac{1}{\partial z} \partial . \frac{1}{R} . x^{-\alpha} \int \frac{\partial x}{x} \int x^\alpha X \partial x$$
$$+ \frac{1}{2 \partial z^2} \partial \partial . \frac{1}{R} . x^{-\alpha} \int x^\alpha X \partial x,$$

si modo in quantitate $R = \frac{P}{(\alpha + z)^3}$ ubique scribatur $z = -\alpha$.

Corollarium 1.

1272. Methodus in solutione hujus problematis adhibita facile ad quotcunque factores aequales extendi potest. Si enim fuerit $(\alpha + z)^m$ factor quantitatis P, atque in hac fractione $\frac{(\alpha + z)^m}{P}$ suisque differentialibus, postquam fuerint evoluta, ponatur $z = -\alpha$, partes integralis inde natae ita se habebunt

Factor quant. P	$\alpha + z$	$(\alpha + z)^2$	$(\alpha + z)^3$
Pars integralis.	$\frac{\alpha + z}{P} x^{-\alpha} \int x^\alpha X \partial x$	$\frac{(\alpha+z)^2}{P} x^{-\alpha} \int \frac{\partial x}{x} \int x^\alpha X \partial x$	$\frac{(\alpha+z)^3}{P} x^{-\alpha} \int \frac{\partial x}{x} \int \frac{\partial x}{x} \int x^\alpha X \partial x$
		$\frac{1}{\partial z} \partial . \frac{(\alpha+z)^2}{P} . x^{-\alpha} \int x^\alpha X \partial x$	$\frac{1}{\partial z} \partial . \frac{(\alpha+z)^3}{P} . x^{-\alpha} \int \frac{\partial x}{x} \int x^\alpha X \partial x$
			$\frac{1}{2 \partial z^2} \partial \partial . \frac{(\alpha+z)^3}{P} . x^{-\alpha} \int x^\alpha X \partial x.$

Corollarium 2.

1273. Si fuerint duo pluresve factores duplices inter se aequales, sumtis

$$\alpha = f(\cos. \theta + \sqrt{-1} . \sin. \theta) \text{ et}$$
$$\beta = f(\cos. \theta - \sqrt{-1} . \sin. \theta),$$

partes pro $(x+z)^2$ et $(\beta+z)^2$ seorsim evolutae methodo supra adhibita non difficulter conjungentur, et ad realitatem reducentur.

Scholion.

1274. Simili methodo, qua hoc caput est pertractatum, in evolutione capitis III. hujus sectionis uti oportebat, neque tum ullum periculum in errores prolabendi fuisset pertimescendum. Superfluum autem nunc foret, errores ibi commissos hic emendare, cum non solum methodus plane esset eadem, sed etiam aequatio hic tractata facile in formam ibi consideratam transmutari queat et vicissim. Quodsi enim in aequatione capitis III.

$$X = Ay + \frac{B\partial y}{\partial x} + \frac{C\partial\partial y}{\partial x^2} + \frac{D\partial^3 y}{\partial x^3} + \frac{E\partial^4 y}{\partial x^4} + \text{etc.}$$

statuatur $x = lv$, ut sit $\partial x = \frac{\partial v}{v}$, functio autem X abeat in functionem ipsius v quae sit V, proveniet aequatio ejus formae quam hic tractavimus. Dum autem ibi elementum ∂x pro constanti est habitum, ad hanc conditionem exuendam ponamus

$$\partial y = p\,\partial x,\ \partial p = q\,\partial x,\ \partial q = r\,\partial x,\ \partial q = s\,\partial x,\ \text{etc.}$$

ut haec aequatio resultet

$$X = V = Ay + Bp + Cq + Dr + Es + Ft + \text{etc.}$$

Nunc autem ob $\partial x = \frac{\partial v}{v}$ adipiscimur, elemento ∂v constante sumto

$$p = \frac{\partial y}{\partial x} = \frac{v\partial y}{\partial v},$$

$$q = \frac{\partial p}{\partial x} = \frac{vv\partial\partial y}{\partial v^2} + \frac{v\partial y}{\partial v},$$

$$r = \frac{\partial q}{\partial x} = \frac{v^3\partial^3 y}{\partial v^3} + \frac{3v^2\partial\partial y}{\partial v^2} + \frac{v\partial y}{\partial v},$$

$$s = \frac{\partial r}{\partial x} = \frac{v^4\partial^4 y}{\partial v^4} + \frac{6v^3\partial^3 y}{\partial v^3} + \frac{7vv\partial\partial y}{\partial v^2} + \frac{v\partial y}{\partial v},$$

$$t = \frac{\partial s}{\partial x} = \frac{v^5\partial^5 y}{\partial v^5} + \frac{10v^4\partial^4 y}{\partial v^4} + \frac{25v^3\partial^3 y}{\partial v^3} + \frac{15vv\partial\partial y}{\partial v^2} + \frac{v\partial y}{\partial v},\ \text{etc.}$$

Quare aequatio inter v et y erit haec:

$$
V = Ay + \begin{array}{l} \dfrac{Bv\partial y}{\partial v} \\ +C \\ +D \\ +E \\ +F \end{array} + \begin{array}{l} \dfrac{Cvv\partial\partial y}{\partial v^2} \\ +3D \\ +7E \\ +15F \\ \end{array} + \begin{array}{l} \dfrac{Dv^3\partial^3 y}{\partial v^3} \\ +6E \\ +25F \\ \\ \end{array} + \begin{array}{l} \dfrac{Ev^4\partial^4 y}{\partial v^4} \\ +10F \\ \\ \\ \end{array} + \begin{array}{l} \dfrac{Fv^5\partial^5 y}{\partial v^5} \\ \\ \\ \\ \end{array} + \text{etc.}
$$

cujus integrationem hic docuimus. Imprimis autem notandum est quantitatem algebraicam P hinc formandam

$$
\begin{aligned}
P = A &+ (B + C + D + E + F)(z - 1) \\
&+ (C + 3D + 7E + 15F)(z - 1)(z - 2) \\
&+ (D + 6E + 25F)(z - 1)(z - 2)(z - 3) + \text{etc.}
\end{aligned}
$$

ad hanc formam reduci

$$
\begin{aligned}
P = A + B(z - 1) + C(z - 1)^2 + D(z - 1)^3 + E(z - 1)^4 \\
+ F(z - 1)^5 + \text{etc.}
\end{aligned}
$$

quae quantitas algebraica ab illa, qua in capite III. ad integrationem sumus usi, hoc tantum differt, quod ibi littera z id quod hic formula $z - 1$ expressimus; ex quo etiam ambarum integratio facillime altera ad alteram reducitur.

Conclusio libri primi.

1275. Atque haec fere sunt, quae ad librum primum de calculo integrali pertinere sunt visa, ubi methodum tradere institui, functiones unius variabilis ex data quacunque differentialium cujusque ordinis relatione investigandi; quod opus mihi equidem ita pertractasse videor, ut vix quicquam eorum, quae adhuc de hoc argumento ab aliis sunt inventa ab aliis sunt inventa et in medium allata, sit praetermissum.